CHAPTER 2 LIMITS AND CONTINUITY

2.1 RATES OF CHANGE AND TANGENTS TO CURVES

1. (a) $\frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2} = \frac{28 - 9}{1} = 19$ (b) $\frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1$

2. (a) $\frac{\Delta g}{\Delta x} = \frac{g(1) - g(-1)}{1 - (-1)} = \frac{1 - 1}{2} = 0$ (b) $\frac{\Delta g}{\Delta x} = \frac{g(0) - g(-2)}{0 - (-2)} = \frac{0 - 4}{2} = -2$

3. (a) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{3\pi}{4}\right) - h\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-1 - 1}{\frac{\pi}{2}} = -\frac{4}{\pi}$ (b) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{\pi}{2}\right) - h\left(\frac{\pi}{6}\right)}{\frac{\pi}{2} - \frac{\pi}{6}} = \frac{0 - \sqrt{3}}{\frac{\pi}{3}} = \frac{-3\sqrt{3}}{\pi}$

4. (a) $\frac{\Delta g}{\Delta t} = \frac{g(\pi) - g(0)}{\pi - 0} = \frac{(2 - 1) - (2 + 1)}{\pi - 0} = -\frac{2}{\pi}$ (b) $\frac{\Delta g}{\Delta t} = \frac{g(\pi) - g(-\pi)}{\pi - (-\pi)} = \frac{(2 - 1) - (2 - 1)}{2\pi} = 0$

5. $\frac{\Delta R}{\Delta \theta} = \frac{R(2) - R(0)}{2 - 0} = \frac{\sqrt{8 + 1} - \sqrt{1}}{2} = \frac{3 - 1}{2} = 1$

6. $\frac{\Delta P}{\Delta \theta} = \frac{P(2) - P(1)}{2 - 1} = \frac{(8 - 16 + 10) - (1 - 4 + 5)}{1} = 2 - 2 = 0$

7. (a) $\frac{\Delta y}{\Delta x} = \frac{\left((2 + h)^2 - 3\right) - \left(2^2 - 3\right)}{h} = \frac{4 + 4h + h^2 - 3 - 1}{h} = \frac{4h + h^2}{h} = 4 + h$. As $h \to 0$, $4 + h \to 4 \Rightarrow$ at $P(2, 1)$ the slope is 4.
   (b) $y - 1 = 4(x - 2) \Rightarrow y - 1 = 4x - 8 \Rightarrow y = 4x - 7$

8. (a) $\frac{\Delta y}{\Delta x} = \frac{\left(5 - (1 + h)^2\right) - \left(5 - 1^2\right)}{h} = \frac{5 - 1 - 2h - h^2 - 4}{h} = \frac{-2h - h^2}{h} = -2 - h$. As $h \to 0$, $-2 - h \to -2 \Rightarrow$ at $P(1, 4)$ the slope is -2.
   (b) $y - 4 = (-2)(x - 1) \Rightarrow y - 4 = -2x + 2 \Rightarrow y = -2x + 6$

9. (a) $\frac{\Delta y}{\Delta x} = \frac{\left((2 + h)^2 - 2(2 + h) - 3\right) - \left(2^2 - 2(2) - 3\right)}{h} = \frac{4 + 4h + h^2 - 4 - 2h - 3 - (-3)}{h} = \frac{2h + h^2}{h} = 2 + h$. As $h \to 0$, $2 + h \to 2 \Rightarrow$ at $P(2, -3)$ the slope is 2.
   (b) $y - (-3) = 2(x - 2) \Rightarrow y + 3 = 2x - 4 \Rightarrow y = 2x - 7$.

10. (a) $\frac{\Delta y}{\Delta x} = \frac{\left((1 + h)^2 - 4(1 + h)\right) - \left(1^2 - 4(1)\right)}{h} = \frac{1 + 2h + h^2 - 4 - 4h - (-3)}{h} = \frac{h^2 - 2h}{h} = h - 2$. As $h \to 0$, $h - 2 \to -2 \Rightarrow$ at $P(1, -3)$ the slope is -2.
   (b) $y - (-3) = (-2)(x - 1) \Rightarrow y + 3 = -2x + 2 \Rightarrow y = -2x - 1$.

11. (a) $\frac{\Delta y}{\Delta x} = \frac{(2 + h)^3 - 2^3}{h} = \frac{8 + 12h + 4h^2 + h^3 - 8}{h} = \frac{12h + 4h^2 + h^3}{h} = 12 + 4h + h^2$. As $h \to 0$, $12 + 4h + h^2 \to 12$, $\Rightarrow$ at $P(2, 8)$ the slope is 12.
   (b) $y - 8 = 12(x - 2) \Rightarrow y - 8 = 12x - 24 \Rightarrow y = 12x - 16$.

12. (a) $\frac{\Delta y}{\Delta x} = \frac{2 - (1 + h)^3 - (2 - 1^3)}{h} = \frac{2 - 1 - 3h - 3h^2 - h^3 - 1}{h} = \frac{-3h - 3h^2 - h^3}{h} = -3 - 3h - h^2$. As $h \to 0$, $-3 - 3h - h^2 \to -3$, $\Rightarrow$ at $P(1, 1)$ the slope is -3.
   (b) $y - 1 = (-3)(x - 1) \Rightarrow y - 1 = -3x + 3 \Rightarrow y = -3x + 4$.

13. (a) $\frac{\Delta y}{\Delta x} = \frac{(1 + h)^3 - 12(1 + h) - \left(1^3 - 12(1)\right)}{h} = \frac{1 + 3h + 3h^2 + h^3 - 12 - 12h - (-11)}{h} = \frac{-9h + 3h^2 + h^3}{h} = -9 + 3h + h^2$. As $h \to 0$, $-9 + 3h + h^2 \to -9 \Rightarrow$ at $P(1, -11)$ the slope is -9.
   (b) $y - (-11) = (-9)(x - 1) \Rightarrow y + 11 = -9x + 9 \Rightarrow y = -9x - 2$.

14. (a) $\frac{\Delta y}{\Delta x} = \frac{(2+h)^3 - 3(2+h)^2 + 4 - (2^3 - 3(2)^2 + 4)}{h} = \frac{8 + 12h + 6h^2 + h^3 - 12 - 12h - 3h^2 + 4 - 0}{h} = \frac{3h^2 + h^3}{h} = 3h + h^2$. As $h \to 0$, $3h + h^2 \to 0 \Rightarrow$ at P(2, 0) the slope is 0.

(b) $y - 0 = 0(x - 2) \Rightarrow y = 0$.

15. (a)

Q	Slope of PQ $= \frac{\Delta p}{\Delta t}$
$Q_1(10, 225)$	$\frac{650 - 225}{20 - 10} = 42.5$ m/sec
$Q_2(14, 375)$	$\frac{650 - 375}{20 - 14} = 45.83$ m/sec
$Q_3(16.5, 475)$	$\frac{650 - 475}{20 - 16.5} = 50.00$ m/sec
$Q_4(18, 550)$	$\frac{650 - 550}{20 - 18} = 50.00$ m/sec

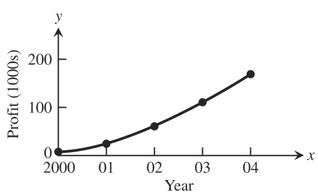
(b) At $t = 20$, the sportscar was traveling approximately 50 m/sec or 180 km/h.

16. (a)

Q	Slope of PQ $= \frac{\Delta p}{\Delta t}$
$Q_1(5, 20)$	$\frac{80 - 20}{10 - 5} = 12$ m/sec
$Q_2(7, 39)$	$\frac{80 - 39}{10 - 7} = 13.7$ m/sec
$Q_3(8.5, 58)$	$\frac{80 - 58}{10 - 8.5} = 14.7$ m/sec
$Q_4(9.5, 72)$	$\frac{80 - 72}{10 - 9.5} = 16$ m/sec

(b) Approximately 16 m/sec

17. (a)



(b) $\frac{\Delta p}{\Delta t} = \frac{174 - 62}{2004 - 2002} = \frac{112}{2} = 56$ thousand dollars per year

(c) The average rate of change from 2001 to 2002 is $\frac{\Delta p}{\Delta t} = \frac{62 - 27}{20022 - 2001} = 35$ thousand dollars per year.

The average rate of change from 2002 to 2003 is $\frac{\Delta p}{\Delta t} = \frac{111 - 62}{2003 - 2002} = 49$ thousand dollars per year.

So, the rate at which profits were changing in 2002 is approximatley $\frac{1}{2}(35 + 49) = 42$ thousand dollars per year.

18. (a) $F(x) = (x + 2)/(x - 2)$

x	1.2	1.1	1.01	1.001	1.0001	1
F(x)	-4.0	$-3.\overline{4}$	$-3.\overline{04}$	$-3.\overline{004}$	$-3.\overline{0004}$	-3

$\frac{\Delta F}{\Delta x} = \frac{-4.0 - (-3)}{1.2 - 1} = -5.0$; $\frac{\Delta F}{\Delta x} = \frac{-3.\overline{4} - (-3)}{1.1 - 1} = -4.\overline{4}$;

$\frac{\Delta F}{\Delta x} = \frac{-3.\overline{04} - (-3)}{1.01 - 1} = -4.\overline{04}$; $\frac{\Delta F}{\Delta x} = \frac{-3.\overline{004} - (-3)}{1.001 - 1} = -4.\overline{004}$;

$\frac{\Delta F}{\Delta x} = \frac{-3.\overline{0004} - (-3)}{1.0001 - 1} = -4.\overline{0004}$;

(b) The rate of change of F(x) at $x = 1$ is -4.

19. (a) $\frac{\Delta g}{\Delta x} = \frac{g(2) - g(1)}{2 - 1} = \frac{\sqrt{2} - 1}{2 - 1} \approx 0.414213$ $\frac{\Delta g}{\Delta x} = \frac{g(1.5) - g(1)}{1.5 - 1} = \frac{\sqrt{1.5} - 1}{0.5} \approx 0.449489$

$\frac{\Delta g}{\Delta x} = \frac{g(1 + h) - g(1)}{(1 + h) - 1} = \frac{\sqrt{1 + h} - 1}{h}$

(b) $g(x) = \sqrt{x}$

$1 + h$	1.1	1.01	1.001	1.0001	1.00001	1.000001
$\sqrt{1 + h}$	1.04880	1.004987	1.0004998	1.0000499	1.000005	1.0000005
$\left(\sqrt{1 + h} - 1\right)/h$	0.4880	0.4987	0.4998	0.499	0.5	0.5

(c) The rate of change of g(x) at $x = 1$ is 0.5.

(d) The calculator gives $\lim_{h \to 0} \frac{\sqrt{1+h}-1}{h} = \frac{1}{2}$.

20. (a) i) $\frac{f(3)-f(2)}{3-2} = \frac{\frac{1}{3}-\frac{1}{2}}{1} = \frac{\frac{-1}{6}}{1} = -\frac{1}{6}$

ii) $\frac{f(T)-f(2)}{T-2} = \frac{\frac{1}{T}-\frac{1}{2}}{T-2} = \frac{\frac{2}{2T}-\frac{T}{2T}}{T-2} = \frac{2-T}{2T(T-2)} = \frac{2-T}{-2T(2-T)} = -\frac{1}{2T}, T \neq 2$

(b)

T	2.1	2.01	2.001	2.0001	2.00001	2.000001
f(T)	0.476190	0.497512	0.499750	0.4999750	0.499997	0.499999
(f(T) − f(2))/(T − 2)	−0.2381	−0.2488	−0.2500	−0.2500	−0.2500	−0.2500

(c) The table indicates the rate of change is -0.25 at $t = 2$.

(d) $\lim_{T \to 2} \left(\frac{1}{-2T}\right) = -\frac{1}{4}$

NOTE: Answers will vary in Exercises 21 and 22.

21. (a) $[0, 1]$: $\frac{\Delta s}{\Delta t} = \frac{15-0}{1-0} = 15$ mph; $[1, 2.5]$: $\frac{\Delta s}{\Delta t} = \frac{20-15}{2.5-1} = \frac{10}{3}$ mph; $[2.5, 3.5]$: $\frac{\Delta s}{\Delta t} = \frac{30-20}{3.5-2.5} = 10$ mph

(b) At $P\left(\frac{1}{2}, 7.5\right)$: Since the portion of the graph from $t = 0$ to $t = 1$ is nearly linear, the instantaneous rate of change will be almost the same as the average rate of change, thus the instantaneous speed at $t = \frac{1}{2}$ is $\frac{15-7.5}{1-0.5} = 15$ mi/hr.

At $P(2, 20)$: Since the portion of the graph from $t = 2$ to $t = 2.5$ is nearly linear, the instantaneous rate of change will be nearly the same as the average rate of change, thus $v = \frac{20-20}{2.5-2} = 0$ mi/hr. For values of t less than 2, we have

Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$
$Q_1(1, 15)$	$\frac{15-20}{1-2} = 5$ mi/hr
$Q_2(1.5, 19)$	$\frac{19-20}{1.5-2} = 2$ mi/hr
$Q_3(1.9, 19.9)$	$\frac{19.9-20}{1.9-2} = 1$ mi/hr

Thus, it appears that the instantaneous speed at $t = 2$ is 0 mi/hr.

At $P(3, 22)$:

Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$	Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-22}{4-3} = 13$ mi/hr	$Q_1(2, 20)$	$\frac{20-22}{2-3} = 2$ mi/hr
$Q_2(3.5, 30)$	$\frac{30-22}{3.5-3} = 16$ mi/hr	$Q_2(2.5, 20)$	$\frac{20-22}{2.5-3} = 4$ mi/hr
$Q_3(3.1, 23)$	$\frac{23-22}{3.1-3} = 10$ mi/hr	$Q_3(2.9, 21.6)$	$\frac{21.6-22}{2.9-3} = 4$ mi/hr

Thus, it appears that the instantaneous speed at $t = 3$ is about 7 mi/hr.

(c) It appears that the curve is increasing the fastest at $t = 3.5$. Thus for $P(3.5, 30)$

Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$	Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-30}{4-3.5} = 10$ mi/hr	$Q_1(3, 22)$	$\frac{22-30}{3-3.5} = 16$ mi/hr
$Q_2(3.75, 34)$	$\frac{34-30}{3.75-3.5} = 16$ mi/hr	$Q_2(3.25, 25)$	$\frac{25-30}{3.25-3.5} = 20$ mi/hr
$Q_3(3.6, 32)$	$\frac{32-30}{3.6-3.5} = 20$ mi/hr	$Q_3(3.4, 28)$	$\frac{28-30}{3.4-3.5} = 20$ mi/hr

Thus, it appears that the instantaneous speed at $t = 3.5$ is about 20 mi/hr.

22. (a) $[0, 3]$: $\frac{\Delta A}{\Delta t} = \frac{10-15}{3-0} \approx -1.67\ \frac{\text{gal}}{\text{day}}$; $[0, 5]$: $\frac{\Delta A}{\Delta t} = \frac{3.9-15}{5-0} \approx -2.2\ \frac{\text{gal}}{\text{day}}$; $[7, 10]$: $\frac{\Delta A}{\Delta t} = \frac{0-1.4}{10-7} \approx -0.5\ \frac{\text{gal}}{\text{day}}$

(b) At $P(1, 14)$:

Q	Slope of PQ $= \frac{\Delta A}{\Delta t}$	Q	Slope of PQ $= \frac{\Delta A}{\Delta t}$
$Q_1(2, 12.2)$	$\frac{12.2-14}{2-1} = -1.8$ gal/day	$Q_1(0, 15)$	$\frac{15-14}{0-1} = -1$ gal/day
$Q_2(1.5, 13.2)$	$\frac{13.2-14}{1.5-1} = -1.6$ gal/day	$Q_2(0.5, 14.6)$	$\frac{14.6-14}{0.5-1} = -1.2$ gal/day
$Q_3(1.1, 13.85)$	$\frac{13.85-14}{1.1-1} = -1.5$ gal/day	$Q_3(0.9, 14.86)$	$\frac{14.86-14}{0.9-1} = -1.4$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is about -1.45 gal/day.

At $P(4, 6)$:

Q	Slope of PQ $= \frac{\Delta A}{\Delta t}$
$Q_1(5, 3.9)$	$\frac{3.9-6}{5-4} = -2.1$ gal/day
$Q_2(4.5, 4.8)$	$\frac{4.8-6}{4.5-4} = -2.4$ gal/day
$Q_3(4.1, 5.7)$	$\frac{5.7-6}{4.1-4} = -3$ gal/day

Q	Slope of PQ $= \frac{\Delta A}{\Delta t}$
$Q_1(3, 10)$	$\frac{10-6}{3-4} = -4$ gal/day
$Q_2(3.5, 7.8)$	$\frac{7.8-6}{3.5-4} = -3.6$ gal/day
$Q_3(3.9, 6.3)$	$\frac{6.3-6}{3.9-4} = -3$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is -3 gal/day.

At P(8, 1):

Q	Slope of PQ $= \frac{\Delta A}{\Delta t}$
$Q_1(9, 0.5)$	$\frac{0.5-1}{9-8} = -0.5$ gal/day
$Q_2(8.5, 0.7)$	$\frac{0.7-1}{8.5-8} = -0.6$ gal/day
$Q_3(8.1, 0.95)$	$\frac{0.95-1}{8.1-8} = -0.5$ gal/day

Q	Slope of PQ $= \frac{\Delta A}{\Delta t}$
$Q_1(7, 1.4)$	$\frac{1.4-1}{7-8} = -0.6$ gal/day
$Q_2(7.5, 1.3)$	$\frac{1.3-1}{7.5-8} = -0.6$ gal/day
$Q_3(7.9, 1.04)$	$\frac{1.04-1}{7.9-8} = -0.6$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is -0.55 gal/day.

(c) It appears that the curve (the consumption) is decreasing the fastest at $t = 3.5$. Thus for P(3.5, 7.8)

Q	Slope of PQ $= \frac{\Delta A}{\Delta t}$
$Q_1(4.5, 4.8)$	$\frac{4.8-7.8}{4.5-3.5} = -3$ gal/day
$Q_2(4, 6)$	$\frac{6-7.8}{4-3.5} = -3.6$ gal/day
$Q_3(3.6, 7.4)$	$\frac{7.4-7.8}{3.6-3.5} = -4$ gal/day

Q	Slope of PQ $= \frac{\Delta s}{\Delta t}$
$Q_1(2.5, 11.2)$	$\frac{11.2-7.8}{2.5-3.5} = -3.4$ gal/day
$Q_2(3, 10)$	$\frac{10-7.8}{3-3.5} = -4.4$ gal/day
$Q_3(3.4, 8.2)$	$\frac{8.2-7.8}{3.4-3.5} = -4$ gal/day

Thus, it appears that the rate of consumption at $t = 3.5$ is about -4 gal/day.

2.2 LIMIT OF A FUNCTION AND LIMIT LAWS

1. (a) Does not exist. As x approaches 1 from the right, g(x) approaches 0. As x approaches 1 from the left, g(x) approaches 1. There is no single number L that all the values g(x) get arbitrarily close to as $x \to 1$.
   (b) 1 (c) 0 (d) 0.5

2. (a) 0
   (b) -1
   (c) Does not exist. As t approaches 0 from the left, f(t) approaches -1. As t approaches 0 from the right, f(t) approaches 1. There is no single number L that f(t) gets arbitrarily close to as $t \to 0$.
   (d) -1

3. (a) True (b) True (c) False
   (d) False (e) False (f) True
   (g) True

4. (a) False (b) False (c) True
   (d) True (e) True

5. $\lim\limits_{x \to 0} \frac{x}{|x|}$ does not exist because $\frac{x}{|x|} = \frac{x}{x} = 1$ if $x > 0$ and $\frac{x}{|x|} = \frac{x}{-x} = -1$ if $x < 0$. As x approaches 0 from the left, $\frac{x}{|x|}$ approaches -1. As x approaches 0 from the right, $\frac{x}{|x|}$ approaches 1. There is no single number L that all the function values get arbitrarily close to as $x \to 0$.

6. As x approaches 1 from the left, the values of $\frac{1}{x-1}$ become increasingly large and negative. As x approaches 1 from the right, the values become increasingly large and positive. There is no one number L that all the function values get arbitrarily close to as $x \to 1$, so $\lim\limits_{x \to 1} \frac{1}{x-1}$ does not exist.

7. Nothing can be said about f(x) because the existence of a limit as $x \to x_0$ does not depend on how the function is defined at x_0. In order for a limit to exist, f(x) must be arbitrarily close to a single real number L when x is close enough to x_0. That is, the existence of a limit depends on the values of f(x) for x <u>near</u> x_0, not on the definition of f(x) at x_0 itself.

8. Nothing can be said. In order for $\lim_{x \to 0} f(x)$ to exist, f(x) must close to a single value for x near 0 regardless of the value f(0) itself.

9. No, the definition does not require that f be defined at $x = 1$ in order for a limiting value to exist there. If f(1) is defined, it can be any real number, so we can conclude nothing about f(1) from $\lim_{x \to 1} f(x) = 5$.

10. No, because the existence of a limit depends on the values of f(x) when x is near 1, not on f(1) itself. If $\lim_{x \to 1} f(x)$ exists, its value may be some number other than $f(1) = 5$. We can conclude nothing about $\lim_{x \to 1} f(x)$, whether it exists or what its value is if it does exist, from knowing the value of f(1) alone.

11. $\lim_{x \to -7} (2x + 5) = 2(-7) + 5 = -14 + 5 = -9$

12. $\lim_{x \to 2} (-x^2 + 5x - 2) = -(2)^2 + 5(2) - 2 = -4 + 10 - 2 = 4$

13. $\lim_{t \to 6} 8(t - 5)(t - 7) = 8(6 - 5)(6 - 7) = -8$

14. $\lim_{x \to -2} (x^3 - 2x^2 + 4x + 8) = (-2)^3 - 2(-2)^2 + 4(-2) + 8 = -8 - 8 - 8 + 8 = -16$

15. $\lim_{x \to 2} \frac{x+3}{x+6} = \frac{2+3}{2+6} = \frac{5}{8}$

16. $\lim_{s \to \frac{2}{3}} 3s(2s - 1) = 3\left(\frac{2}{3}\right)\left[2\left(\frac{2}{3}\right) - 1\right] = 2\left(\frac{4}{3} - 1\right) = \frac{2}{3}$

17. $\lim_{x \to -1} 3(2x - 1)^2 = 3(2(-1) - 1)^2 = 3(-3)^2 = 27$

18. $\lim_{y \to 2} \frac{y+2}{y^2+5y+6} = \frac{2+2}{(2)^2+5(2)+6} = \frac{4}{4+10+6} = \frac{4}{20} = \frac{1}{5}$

19. $\lim_{y \to -3} (5 - y)^{4/3} = [5 - (-3)]^{4/3} = (8)^{4/3} = \left((8)^{1/3}\right)^4 = 2^4 = 16$

20. $\lim_{z \to 0} (2z - 8)^{1/3} = (2(0) - 8)^{1/3} = (-8)^{1/3} = -2$

21. $\lim_{h \to 0} \frac{3}{\sqrt{3h+1}+1} = \frac{3}{\sqrt{3(0)+1}+1} = \frac{3}{\sqrt{1}+1} = \frac{3}{2}$

22. $\lim_{h \to 0} \frac{\sqrt{5h+4}-2}{h} = \lim_{h \to 0} \frac{\sqrt{5h+4}-2}{h} \cdot \frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2} = \lim_{h \to 0} \frac{(5h+4)-4}{h\left(\sqrt{5h+4}+2\right)} = \lim_{h \to 0} \frac{5h}{h\left(\sqrt{5h+4}+2\right)} = \lim_{h \to 0} \frac{5}{\sqrt{5h+4}+2}$
$= \frac{5}{\sqrt{4}+2} = \frac{5}{4}$

23. $\lim_{x \to 5} \frac{x-5}{x^2-25} = \lim_{x \to 5} \frac{x-5}{(x+5)(x-5)} = \lim_{x \to 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$

24. $\lim_{x \to -3} \frac{x+3}{x^2+4x+3} = \lim_{x \to -3} \frac{x+3}{(x+3)(x+1)} = \lim_{x \to -3} \frac{1}{x+1} = \frac{1}{-3+1} = -\frac{1}{2}$

25. $\lim_{x \to -5} \frac{x^2+3x-10}{x+5} = \lim_{x \to -5} \frac{(x+5)(x-2)}{x+5} = \lim_{x \to -5} (x-2) = -5-2 = -7$

26. $\lim_{x \to 2} \frac{x^2-7x+10}{x-2} = \lim_{x \to 2} \frac{(x-5)(x-2)}{x-2} = \lim_{x \to 2} (x-5) = 2-5 = -3$

27. $\lim_{t \to 1} \frac{t^2+t-2}{t^2-1} = \lim_{t \to 1} \frac{(t+2)(t-1)}{(t-1)(t+1)} = \lim_{t \to 1} \frac{t+2}{t+1} = \frac{1+2}{1+1} = \frac{3}{2}$

28. $\lim_{t \to -1} \frac{t^2+3t+2}{t^2-t-2} = \lim_{t \to -1} \frac{(t+2)(t+1)}{(t-2)(t+1)} = \lim_{t \to -1} \frac{t+2}{t-2} = \frac{-1+2}{-1-2} = -\frac{1}{3}$

29. $\lim_{x \to -2} \frac{-2x-4}{x^3+2x^2} = \lim_{x \to -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x \to -2} \frac{-2}{x^2} = \frac{-2}{4} = -\frac{1}{2}$

30. $\lim_{y \to 0} \frac{5y^3+8y^2}{3y^4-16y^2} = \lim_{y \to 0} \frac{y^2(5y+8)}{y^2(3y^2-16)} = \lim_{y \to 0} \frac{5y+8}{3y^2-16} = \frac{8}{-16} = -\frac{1}{2}$

31. $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1} = \lim_{x \to 1} \frac{\frac{1-x}{x}}{x-1} = \lim_{x \to 1} \left(\frac{1-x}{x} \cdot \frac{1}{x-1}\right) = \lim_{x \to 1} -\frac{1}{x} = -1$

32. $\lim_{x \to 0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} = \lim_{x \to 1} \frac{\frac{(x+1)+(x-1)}{(x-1)(x+1)}}{x} = \lim_{x \to 1} \left(\frac{2x}{(x-1)(x+1)} \cdot \frac{1}{x}\right) = \lim_{x \to 1} \frac{2}{(x-1)(x+1)} = \frac{2}{-1} = -2$

33. $\lim_{u \to 1} \frac{u^4-1}{u^3-1} = \lim_{u \to 1} \frac{(u^2+1)(u+1)(u-1)}{(u^2+u+1)(u-1)} = \lim_{u \to 1} \frac{(u^2+1)(u+1)}{u^2+u+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$

34. $\lim_{v \to 2} \frac{v^3-8}{v^4-16} = \lim_{v \to 2} \frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} = \lim_{v \to 2} \frac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{4+4+4}{(4)(8)} = \frac{12}{32} = \frac{3}{8}$

35. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x \to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

36. $\lim_{x \to 4} \frac{4x-x^2}{2-\sqrt{x}} = \lim_{x \to 4} \frac{x(4-x)}{2-\sqrt{x}} = \lim_{x \to 4} \frac{x(2+\sqrt{x})(2-\sqrt{x})}{2-\sqrt{x}} = \lim_{x \to 4} x(2+\sqrt{x}) = 4(2+2) = 16$

37. $\lim_{x \to 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x \to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)} = \lim_{x \to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4} = \lim_{x \to 1} \left(\sqrt{x+3}+2\right) = \sqrt{4}+2 = 4$

38. $\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x \to -1} \frac{\left(\sqrt{x^2+8}-3\right)\left(\sqrt{x^2+8}+3\right)}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim_{x \to -1} \frac{(x^2+8)-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$

$= \lim_{x \to -1} \frac{(x+1)(x-1)}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim_{x \to -1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}$

39. $\lim_{x \to 2} \frac{\sqrt{x^2+12}-4}{x-2} = \lim_{x \to 2} \frac{\left(\sqrt{x^2+12}-4\right)\left(\sqrt{x^2+12}+4\right)}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim_{x \to 2} \frac{(x^2+12)-16}{(x-2)\left(\sqrt{x^2+12}+4\right)}$

$= \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim_{x \to 2} \frac{x+2}{\sqrt{x^2+12}+4} = \frac{4}{\sqrt{16}+4} = \frac{1}{2}$

40. $\lim_{x \to -2} \frac{x+2}{\sqrt{x^2+5}-3} = \lim_{x \to -2} \frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{\left(\sqrt{x^2+5}-3\right)\left(\sqrt{x^2+5}+3\right)} = \lim_{x \to -2} \frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{(x^2+5)-9}$

$= \lim_{x \to -2} \frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{(x+2)(x-2)} = \lim_{x \to -2} \frac{\sqrt{x^2+5}+3}{x-2} = \frac{\sqrt{9}+3}{-4} = -\frac{3}{2}$

41. $\lim\limits_{x \to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \lim\limits_{x \to -3} \frac{\left(2-\sqrt{x^2-5}\right)\left(2+\sqrt{x^2-5}\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)} = \lim\limits_{x \to -3} \frac{4-(x^2-5)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$

$= \lim\limits_{x \to -3} \frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)} = \lim\limits_{x \to -3} \frac{(3-x)(3+x)}{(x+3)\left(2+\sqrt{x^2-5}\right)} = \lim\limits_{x \to -3} \frac{3-x}{2+\sqrt{x^2-5}} = \frac{6}{2+\sqrt{4}} = \frac{3}{2}$

42. $\lim\limits_{x \to 4} \frac{4-x}{5-\sqrt{x^2+9}} = \lim\limits_{x \to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{\left(5-\sqrt{x^2+9}\right)\left(5+\sqrt{x^2+9}\right)} = \lim\limits_{x \to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{25-(x^2+9)}$

$= \lim\limits_{x \to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{16-x^2} = \lim\limits_{x \to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{(4-x)(4+x)} = \lim\limits_{x \to 4} \frac{5+\sqrt{x^2+9}}{4+x} = \frac{5+\sqrt{25}}{8} = \frac{5}{4}$

43. $\lim\limits_{x \to 0} (2\sin x - 1) = 2\sin 0 - 1 = 0 - 1 = -1$

44. $\lim\limits_{x \to 0} \sin^2 x = \left(\lim\limits_{x \to 0} \sin x\right)^2 = (\sin 0)^2 = 0^2 = 0$

45. $\lim\limits_{x \to 0} \sec x = \lim\limits_{x \to 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$

46. $\lim\limits_{x \to 0} \tan x = \lim\limits_{x \to 0} \frac{\sin x}{\cos x} = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$

47. $\lim\limits_{x \to 0} \frac{1+x+\sin x}{3\cos x} = \frac{1+0+\sin 0}{3\cos 0} = \frac{1+0+0}{3} = \frac{1}{3}$

48. $\lim\limits_{x \to 0} (x^2-1)(2-\cos x) = (0^2-1)(2-\cos 0) = (-1)(2-1) = (-1)(1) = -1$

49. $\lim\limits_{x \to -\pi} \sqrt{x+4}\,\cos(x+\pi) = \lim\limits_{x \to -\pi} \sqrt{x+4} \cdot \lim\limits_{x \to -\pi} \cos(x+\pi) = \sqrt{-\pi+4} \cdot \cos 0 = \sqrt{4-\pi} \cdot 1 = \sqrt{4-\pi}$

50. $\lim\limits_{x \to 0} \sqrt{7+\sec^2 x} = \sqrt{\lim\limits_{x \to 0} (7+\sec^2 x)} = \sqrt{7+\lim\limits_{x \to 0} \sec^2 x} = \sqrt{7+\sec^2 0} = \sqrt{7+(1)^2} = 2\sqrt{2}$

51. (a) quotient rule
(b) difference and power rules
(c) sum and constant multiple rules

52. (a) quotient rule
(b) power and product rules
(c) difference and constant multiple rules

53. (a) $\lim\limits_{x \to c} f(x)\,g(x) = \left[\lim\limits_{x \to c} f(x)\right]\left[\lim\limits_{x \to c} g(x)\right] = (5)(-2) = -10$

(b) $\lim\limits_{x \to c} 2f(x)\,g(x) = 2\left[\lim\limits_{x \to c} f(x)\right]\left[\lim\limits_{x \to c} g(x)\right] = 2(5)(-2) = -20$

(c) $\lim\limits_{x \to c} [f(x)+3g(x)] = \lim\limits_{x \to c} f(x) + 3\lim\limits_{x \to c} g(x) = 5 + 3(-2) = -1$

(d) $\lim\limits_{x \to c} \frac{f(x)}{f(x)-g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} f(x) - \lim_{x \to c} g(x)} = \frac{5}{5-(-2)} = \frac{5}{7}$

54. (a) $\lim\limits_{x \to 4} [g(x)+3] = \lim\limits_{x \to 4} g(x) + \lim\limits_{x \to 4} 3 = -3+3 = 0$

(b) $\lim\limits_{x \to 4} xf(x) = \lim\limits_{x \to 4} x \cdot \lim\limits_{x \to 4} f(x) = (4)(0) = 0$

(c) $\lim\limits_{x \to 4} [g(x)]^2 = \left[\lim\limits_{x \to 4} g(x)\right]^2 = [-3]^2 = 9$

(d) $\lim\limits_{x \to 4} \frac{g(x)}{f(x)-1} = \frac{\lim_{x \to 4} g(x)}{\lim_{x \to 4} f(x) - \lim_{x \to 4} 1} = \frac{-3}{0-1} = 3$

55. (a) $\lim\limits_{x \to b} [f(x)+g(x)] = \lim\limits_{x \to b} f(x) + \lim\limits_{x \to b} g(x) = 7 + (-3) = 4$

(b) $\lim\limits_{x \to b} f(x) \cdot g(x) = \left[\lim\limits_{x \to b} f(x)\right]\left[\lim\limits_{x \to b} g(x)\right] = (7)(-3) = -21$

(c) $\lim\limits_{x \to b} 4g(x) = \left[\lim\limits_{x \to b} 4\right]\left[\lim\limits_{x \to b} g(x)\right] = (4)(-3) = -12$

(d) $\lim\limits_{x \to b} f(x)/g(x) = \lim\limits_{x \to b} f(x)/\lim\limits_{x \to b} g(x) = \frac{7}{-3} = -\frac{7}{3}$

56. (a) $\lim\limits_{x \to -2} [p(x) + r(x) + s(x)] = \lim\limits_{x \to -2} p(x) + \lim\limits_{x \to -2} r(x) + \lim\limits_{x \to -2} s(x) = 4 + 0 + (-3) = 1$

(b) $\lim\limits_{x \to -2} p(x) \cdot r(x) \cdot s(x) = \left[\lim\limits_{x \to -2} p(x)\right]\left[\lim\limits_{x \to -2} r(x)\right]\left[\lim\limits_{x \to -2} s(x)\right] = (4)(0)(-3) = 0$

(c) $\lim\limits_{x \to -2} [-4p(x) + 5r(x)]/s(x) = \left[-4 \lim\limits_{x \to -2} p(x) + 5 \lim\limits_{x \to -2} r(x)\right] \Big/ \lim\limits_{x \to -2} s(x) = [-4(4) + 5(0)]/-3 = \frac{16}{3}$

57. $\lim\limits_{h \to 0} \frac{(1+h)^2 - 1^2}{h} = \lim\limits_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim\limits_{h \to 0} \frac{h(2+h)}{h} = \lim\limits_{h \to 0} (2 + h) = 2$

58. $\lim\limits_{h \to 0} \frac{(-2+h)^2 - (-2)^2}{h} = \lim\limits_{h \to 0} \frac{4 - 4h + h^2 - 4}{h} = \lim\limits_{h \to 0} \frac{h(h-4)}{h} = \lim\limits_{h \to 0} (h - 4) = -4$

59. $\lim\limits_{h \to 0} \frac{[3(2+h) - 4] - [3(2) - 4]}{h} = \lim\limits_{h \to 0} \frac{3h}{h} = 3$

60. $\lim\limits_{h \to 0} \frac{\left(\frac{1}{-2+h}\right) - \left(\frac{1}{-2}\right)}{h} = \lim\limits_{h \to 0} \frac{\frac{-2}{-2+h} - 1}{-2h} = \lim\limits_{h \to 0} \frac{-2 - (-2+h)}{-2h(-2+h)} = \lim\limits_{h \to 0} \frac{-h}{h(4-2h)} = -\frac{1}{4}$

61. $\lim\limits_{h \to 0} \frac{\sqrt{7+h} - \sqrt{7}}{h} = \lim\limits_{h \to 0} \frac{\left(\sqrt{7+h} - \sqrt{7}\right)\left(\sqrt{7+h} + \sqrt{7}\right)}{h\left(\sqrt{7+h} + \sqrt{7}\right)} = \lim\limits_{h \to 0} \frac{(7+h) - 7}{h\left(\sqrt{7+h} + \sqrt{7}\right)} = \lim\limits_{h \to 0} \frac{h}{h\left(\sqrt{7+h} + \sqrt{7}\right)} = \lim\limits_{h \to 0} \frac{1}{\sqrt{7+h} + \sqrt{7}}$
$= \frac{1}{2\sqrt{7}}$

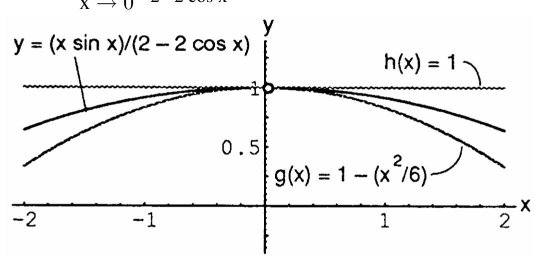
62. $\lim\limits_{h \to 0} \frac{\sqrt{3(0+h)+1} - \sqrt{3(0)+1}}{h} = \lim\limits_{h \to 0} \frac{\left(\sqrt{3h+1} - 1\right)\left(\sqrt{3h+1} + 1\right)}{h\left(\sqrt{3h+1} + 1\right)} = \lim\limits_{h \to 0} \frac{(3h+1) - 1}{h\left(\sqrt{3h+1} + 1\right)} = \lim\limits_{h \to 0} \frac{3h}{h\left(\sqrt{3h+1} + 1\right)}$
$= \lim\limits_{h \to 0} \frac{3}{\sqrt{3h+1} + 1} = \frac{3}{2}$

63. $\lim\limits_{x \to 0} \sqrt{5 - 2x^2} = \sqrt{5 - 2(0)^2} = \sqrt{5}$ and $\lim\limits_{x \to 0} \sqrt{5 - x^2} = \sqrt{5 - (0)^2} = \sqrt{5}$; by the sandwich theorem, $\lim\limits_{x \to 0} f(x) = \sqrt{5}$

64. $\lim\limits_{x \to 0} (2 - x^2) = 2 - 0 = 2$ and $\lim\limits_{x \to 0} 2 \cos x = 2(1) = 2$; by the sandwich theorem, $\lim\limits_{x \to 0} g(x) = 2$

65. (a) $\lim\limits_{x \to 0} \left(1 - \frac{x^2}{6}\right) = 1 - \frac{0}{6} = 1$ and $\lim\limits_{x \to 0} 1 = 1$; by the sandwich theorem, $\lim\limits_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = 1$

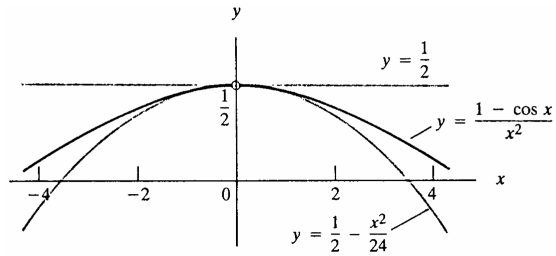
(b) For $x \neq 0$, $y = (x \sin x)/(2 - 2\cos x)$ lies between the other two graphs in the figure, and the graphs converge as $x \to 0$.



66. (a) $\lim\limits_{x \to 0} \left(\frac{1}{2} - \frac{x^2}{24}\right) = \lim\limits_{x \to 0} \frac{1}{2} - \lim\limits_{x \to 0} \frac{x^2}{24} = \frac{1}{2} - 0 = \frac{1}{2}$ and $\lim\limits_{x \to 0} \frac{1}{2} = \frac{1}{2}$; by the sandwich theorem,

$\lim\limits_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$.

(b) For all $x \neq 0$, the graph of $f(x) = (1 - \cos x)/x^2$ lies between the line $y = \frac{1}{2}$ and the parabola $y = \frac{1}{2} - x^2/24$, and the graphs converge as $x \to 0$.



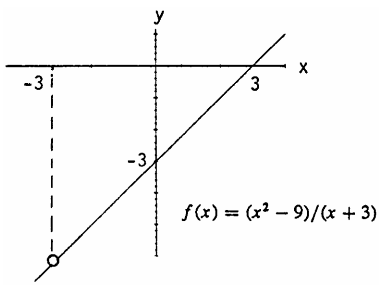
67. (a) $f(x) = (x^2 - 9)/(x + 3)$

x	−3.1	−3.01	−3.001	−3.0001	−3.00001	−3.000001
f(x)	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001

x	−2.9	−2.99	−2.999	−2.9999	−2.99999	−2.999999
f(x)	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999

The estimate is $\lim\limits_{x \to -3} f(x) = -6$.

(b)

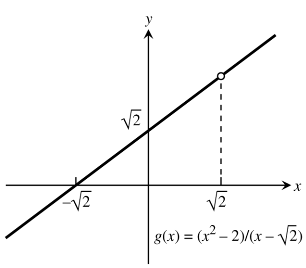


(c) $f(x) = \frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3} = x - 3$ if $x \neq -3$, and $\lim\limits_{x \to -3} (x - 3) = -3 - 3 = -6$.

68. (a) $g(x) = (x^2 - 2)/\left(x - \sqrt{2}\right)$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
g(x)	2.81421	2.82421	2.82821	2.828413	2.828423	2.828426

(b)



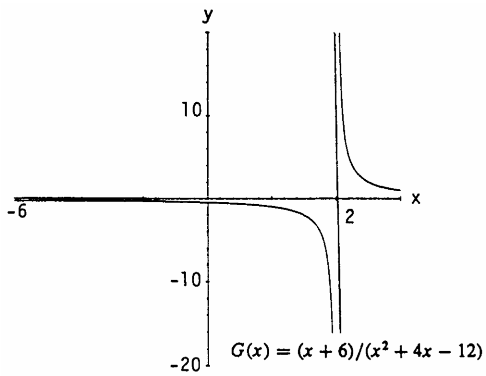
(c) $g(x) = \frac{x^2-2}{x-\sqrt{2}} = \frac{\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)}{\left(x-\sqrt{2}\right)} = x + \sqrt{2}$ if $x \neq \sqrt{2}$, and $\lim\limits_{x \to \sqrt{2}} \left(x + \sqrt{2}\right) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$.

69. (a) $G(x) = (x + 6)/\left(x^2 + 4x - 12\right)$

x	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999
G(x)	−.126582	−.1251564	−.1250156	−.1250015	−.1250001	−.1250000

x	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001
G(x)	−.123456	−.124843	−.124984	−.124998	−.124999	−.124999

(b)



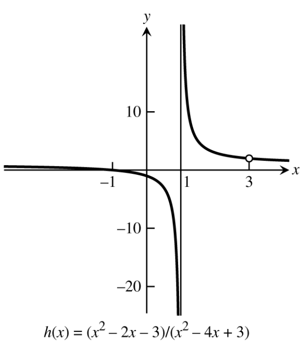
(c) $G(x) = \frac{x+6}{(x^2+4x-12)} = \frac{x+6}{(x+6)(x-2)} = \frac{1}{x-2}$ if $x \neq -6$, and $\lim_{x \to -6} \frac{1}{x-2} = \frac{1}{-6-2} = -\frac{1}{8} = -0.125$.

70. (a) $h(x) = (x^2 - 2x - 3) / (x^2 - 4x + 3)$

x	2.9	2.99	2.999	2.9999	2.99999	2.999999
h(x)	2.052631	2.005025	2.000500	2.000050	2.000005	2.0000005

x	3.1	3.01	3.001	3.0001	3.00001	3.000001
h(x)	1.952380	1.995024	1.999500	1.999950	1.999995	1.999999

(b)



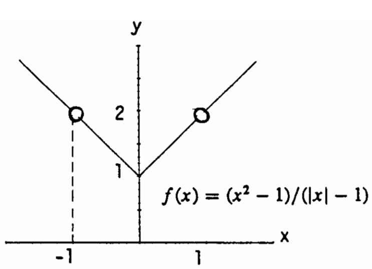
(c) $h(x) = \frac{x^2-2x-3}{x^2-4x+3} = \frac{(x-3)(x+1)}{(x-3)(x-1)} = \frac{x+1}{x-1}$ if $x \neq 3$, and $\lim_{x \to 3} \frac{x+1}{x-1} = \frac{3+1}{3-1} = \frac{4}{2} = 2$.

71. (a) $f(x) = (x^2 - 1) / (|x| - 1)$

x	−1.1	−1.01	−1.001	−1.0001	−1.00001	−1.000001
f(x)	2.1	2.01	2.001	2.0001	2.00001	2.000001

x	−.9	−.99	−.999	−.9999	−.99999	−.999999
f(x)	1.9	1.99	1.999	1.9999	1.99999	1.999999

(b)



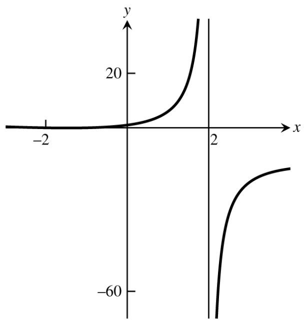
(c) $f(x) = \frac{x^2-1}{|x|-1} = \begin{cases} \frac{(x+1)(x-1)}{x-1} = x+1, & x \geq 0 \text{ and } x \neq 1 \\ \frac{(x+1)(x-1)}{-(x+1)} = 1-x, & x < 0 \text{ and } x \neq -1 \end{cases}$, and $\lim_{x \to -1} (1-x) = 1-(-1) = 2$.

72. (a) $F(x) = (x^2 + 3x + 2)/(2 - |x|)$

x	-2.1	-2.01	-2.001	-2.0001	-2.00001	-2.000001
F(x)	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001

x	-1.9	-1.99	-1.999	-1.9999	-1.99999	-1.999999
F(x)	$-.9$	$-.99$	$-.999$	$-.9999$	$-.99999$	$-.999999$

(b)



$F(x) = (x^2 + 3x + 2)/(2 - |x|)$

(c) $F(x) = \frac{x^2+3x+2}{2-|x|} = \begin{cases} \frac{(x+2)(x+1)}{2-x}, & x \ge 0 \\ \frac{(x+2)(x+1)}{2+x} = x+1, & x < 0 \text{ and } x \ne -2 \end{cases}$, and $\lim_{x \to -2} (x+1) = -2 + 1 = -1$.

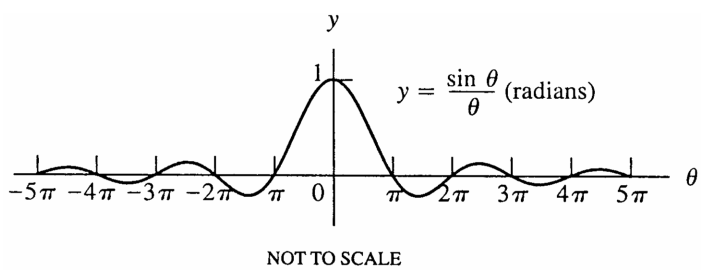
73. (a) $g(\theta) = (\sin\theta)/\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

θ	$-.1$	$-.01$	$-.001$	$-.0001$	$-.00001$	$-.000001$
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim_{\theta \to 0} g(\theta) = 1$

(b)



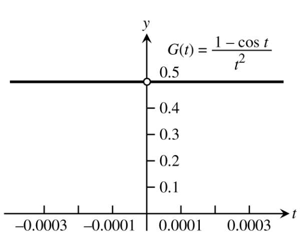
74. (a) $G(t) = (1 - \cos t)/t^2$

t	.1	.01	.001	.0001	.00001	.000001
G(t)	.499583	.499995	.499999	.5	.5	.5

t	$-.1$	$-.01$	$-.001$	$-.0001$	$-.00001$	$-.000001$
G(t)	.499583	.499995	.499999	.5	.5	.5

$\lim_{t \to 0} G(t) = 0.5$

(b)



Graph is NOT TO SCALE

75. $\lim_{x \to c} f(x)$ exists at those points c where $\lim_{x \to c} x^4 = \lim_{x \to c} x^2$. Thus, $c^4 = c^2 \Rightarrow c^2(1 - c^2) = 0 \Rightarrow c = 0, 1$, or -1.

Moreover, $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0$ and $\lim_{x \to -1} f(x) = \lim_{x \to 1} f(x) = 1$.

76. Nothing can be concluded about the values of f, g, and h at $x = 2$. Yes, f(2) could be 0. Since the conditions of the sandwich theorem are satisfied, $\lim_{x \to 2} f(x) = -5 \neq 0$.

77. $1 = \lim_{x \to 4} \frac{f(x)-5}{x-2} = \frac{\lim_{x\to4} f(x) - \lim_{x\to4} 5}{\lim_{x\to4} x - \lim_{x\to4} 2} = \frac{\lim_{x\to4} f(x) - 5}{4-2} \Rightarrow \lim_{x \to 4} f(x) - 5 = 2(1) \Rightarrow \lim_{x \to 4} f(x) = 2 + 5 = 7.$

78. (a) $1 = \lim_{x \to -2} \frac{f(x)}{x^2} = \frac{\lim_{x\to-2} f(x)}{\lim_{x\to-2} x^2} = \frac{\lim_{x\to-2} f(x)}{4} \Rightarrow \lim_{x \to -2} f(x) = 4.$

(b) $1 = \lim_{x \to -2} \frac{f(x)}{x^2} = \left[\lim_{x \to -2} \frac{f(x)}{x}\right]\left[\lim_{x \to -2} \frac{1}{x}\right] = \left[\lim_{x \to -2} \frac{f(x)}{x}\right]\left(\frac{1}{-2}\right) \Rightarrow \lim_{x \to -2} \frac{f(x)}{x} = -2.$

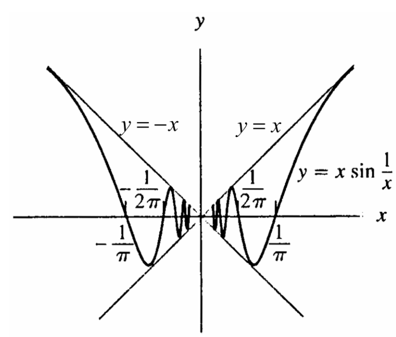
79. (a) $0 = 3 \cdot 0 = \left[\lim_{x \to 2} \frac{f(x)-5}{x-2}\right]\left[\lim_{x \to 2} (x-2)\right] = \lim_{x \to 2} \left[\left(\frac{f(x)-5}{x-2}\right)(x-2)\right] = \lim_{x \to 2} [f(x) - 5] = \lim_{x \to 2} f(x) - 5$
$\Rightarrow \lim_{x \to 2} f(x) = 5.$

(b) $0 = 4 \cdot 0 = \left[\lim_{x \to 2} \frac{f(x)-5}{x-2}\right]\left[\lim_{x \to 2} (x-2)\right] \Rightarrow \lim_{x \to 2} f(x) = 5$ as in part (a).

80. (a) $0 = 1 \cdot 0 = \left[\lim_{x \to 0} \frac{f(x)}{x^2}\right]\left[\lim_{x \to 0} x\right]^2 = \left[\lim_{x \to 0} \frac{f(x)}{x^2}\right]\left[\lim_{x \to 0} x^2\right] = \lim_{x \to 0} \left[\frac{f(x)}{x^2} \cdot x^2\right] = \lim_{x \to 0} f(x)$. That is, $\lim_{x \to 0} f(x) = 0$.

(b) $0 = 1 \cdot 0 = \left[\lim_{x \to 0} \frac{f(x)}{x^2}\right]\left[\lim_{x \to 0} x\right] = \lim_{x \to 0} \left[\frac{f(x)}{x^2} \cdot x\right] = \lim_{x \to 0} \frac{f(x)}{x}$. That is, $\lim_{x \to 0} \frac{f(x)}{x} = 0$.

81. (a) $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

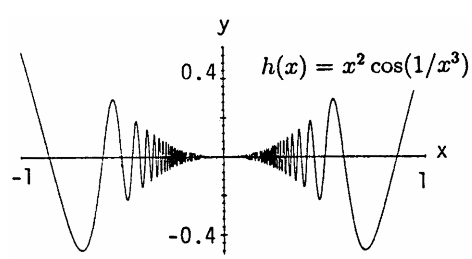


(b) $-1 \le \sin \frac{1}{x} \le 1$ for $x \neq 0$:

$x > 0 \Rightarrow -x \le x \sin \frac{1}{x} \le x \Rightarrow \lim_{x \to 0} x \sin \frac{1}{x} = 0$ by the sandwich theorem;

$x < 0 \Rightarrow -x \ge x \sin \frac{1}{x} \ge x \Rightarrow \lim_{x \to 0} x \sin \frac{1}{x} = 0$ by the sandwich theorem.

82. (a) $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^3}\right) = 0$



(b) $-1 \le \cos\left(\frac{1}{x^3}\right) \le 1$ for $x \ne 0 \Rightarrow -x^2 \le x^2 \cos\left(\frac{1}{x^3}\right) \le x^2 \Rightarrow \lim\limits_{x \to 0} x^2 \cos\left(\frac{1}{x^3}\right) = 0$ by the sandwich theorem since $\lim\limits_{x \to 0} x^2 = 0$.

83-88. Example CAS commands:

<u>Maple</u>:

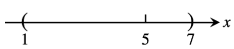
```
f := x -> (x^4 − 16)/(x − 2);
x0 := 2;
plot( f(x), x = x0-1..x0+1, color = black,
      title = "Section 2.2, #83(a)" );
limit( f(x), x = x0 );
```

In Exercise 85, note that the standard cube root, x^(1/3), is not defined for x<0 in many CASs. This can be overcome in Maple by entering the function as f := x -> (surd(x+1, 3) − 1)/x.

<u>Mathematica</u>: (assigned function and values for x0 and h may vary)

```
Clear[f, x]
f[x_]:=(x^3 − x^2 − 5x − 3)/(x + 1)^2
x0= −1; h= 0.1;
Plot[f[x],{x, x0 − h, x0 + h}]
Limit[f[x], x → x0]
```

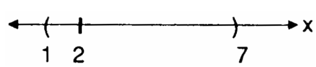
2.3 THE PRECISE DEFINITION OF A LIMIT

1. 

Step 1: $|x-5| < \delta \Rightarrow -\delta < x-5 < \delta \Rightarrow -\delta+5 < x < \delta+5$

Step 2: $\delta + 5 = 7 \Rightarrow \delta = 2$, or $-\delta + 5 = 1 \Rightarrow \delta = 4$.

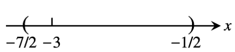
The value of δ which assures $|x-5| < \delta \Rightarrow 1 < x < 7$ is the smaller value, $\delta = 2$.

2. 

Step 1: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$

Step 2: $-\delta + 2 = 1 \Rightarrow \delta = 1$, or $\delta + 2 = 7 \Rightarrow \delta = 5$.

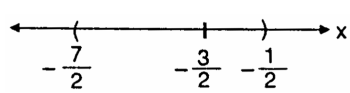
The value of δ which assures $|x-2| < \delta \Rightarrow 1 < x < 7$ is the smaller value, $\delta = 1$.

3. 

Step 1: $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3$

Step 2: $-\delta - 3 = -\frac{7}{2} \Rightarrow \delta = \frac{1}{2}$, or $\delta - 3 = -\frac{1}{2} \Rightarrow \delta = \frac{5}{2}$.

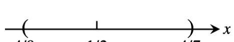
The value of δ which assures $|x-(-3)| < \delta \Rightarrow -\frac{7}{2} < x < -\frac{1}{2}$ is the smaller value, $\delta = \frac{1}{2}$.

4. 

Step 1: $\left|x - \left(-\frac{3}{2}\right)\right| < \delta \Rightarrow -\delta < x + \frac{3}{2} < \delta \Rightarrow -\delta - \frac{3}{2} < x < \delta - \frac{3}{2}$

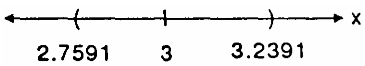
Step 2: $-\delta - \frac{3}{2} = -\frac{7}{2} \Rightarrow \delta = 2$, or $\delta - \frac{3}{2} = -\frac{1}{2} \Rightarrow \delta = 1$.

The value of δ which assures $\left|x - \left(-\frac{3}{2}\right)\right| < \delta \Rightarrow -\frac{7}{2} < x < -\frac{1}{2}$ is the smaller value, $\delta = 1$.

5. 

Step 1: $\left|x - \frac{1}{2}\right| < \delta \Rightarrow -\delta < x - \frac{1}{2} < \delta \Rightarrow -\delta + \frac{1}{2} < x < \delta + \frac{1}{2}$

Step 2: $-\delta + \frac{1}{2} = \frac{4}{9} \Rightarrow \delta = \frac{1}{18}$, or $\delta + \frac{1}{2} = \frac{4}{7} \Rightarrow \delta = \frac{1}{14}$.
The value of δ which assures $\left|x - \frac{1}{2}\right| < \delta \Rightarrow \frac{4}{9} < x < \frac{4}{7}$ is the smaller value, $\delta = \frac{1}{18}$.

6.


Step 1: $|x - 3| < \delta \Rightarrow -\delta < x - 3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3$
Step 2: $-\delta + 3 = 2.7591 \Rightarrow \delta = 0.2409$, or $\delta + 3 = 3.2391 \Rightarrow \delta = 0.2391$.
The value of δ which assures $|x - 3| < \delta \Rightarrow 2.7591 < x < 3.2391$ is the smaller value, $\delta = 0.2391$.

7. Step 1: $|x - 5| < \delta \Rightarrow -\delta < x - 5 < \delta \Rightarrow -\delta + 5 < x < \delta + 5$
Step 2: From the graph, $-\delta + 5 = 4.9 \Rightarrow \delta = 0.1$, or $\delta + 5 = 5.1 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$ in either case.

8. Step 1: $|x - (-3)| < \delta \Rightarrow -\delta < x + 3 < \delta \Rightarrow -\delta - 3 < x < \delta - 3$
Step 2: From the graph, $-\delta - 3 = -3.1 \Rightarrow \delta = 0.1$, or $\delta - 3 = -2.9 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$.

9. Step 1: $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow -\delta + 1 < x < \delta + 1$
Step 2: From the graph, $-\delta + 1 = \frac{9}{16} \Rightarrow \delta = \frac{7}{16}$, or $\delta + 1 = \frac{25}{16} \Rightarrow \delta = \frac{9}{16}$; thus $\delta = \frac{7}{16}$.

10. Step 1: $|x - 3| < \delta \Rightarrow -\delta < x - 3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3$
Step 2: From the graph, $-\delta + 3 = 2.61 \Rightarrow \delta = 0.39$, or $\delta + 3 = 3.41 \Rightarrow \delta = 0.41$; thus $\delta = 0.39$.

11. Step 1: $|x - 2| < \delta \Rightarrow -\delta < x - 2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2$
Step 2: From the graph, $-\delta + 2 = \sqrt{3} \Rightarrow \delta = 2 - \sqrt{3} \approx 0.2679$, or $\delta + 2 = \sqrt{5} \Rightarrow \delta = \sqrt{5} - 2 \approx 0.2361$; thus $\delta = \sqrt{5} - 2$.

12. Step 1: $|x - (-1)| < \delta \Rightarrow -\delta < x + 1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$
Step 2: From the graph, $-\delta - 1 = -\frac{\sqrt{5}}{2} \Rightarrow \delta = \frac{\sqrt{5} - 2}{2} \approx 0.1180$, or $\delta - 1 = -\frac{\sqrt{3}}{2} \Rightarrow \delta = \frac{2 - \sqrt{3}}{2} \approx 0.1340$; thus $\delta = \frac{\sqrt{5} - 2}{2}$.

13. Step 1: $|x - (-1)| < \delta \Rightarrow -\delta < x + 1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$
Step 2: From the graph, $-\delta - 1 = -\frac{16}{9} \Rightarrow \delta = \frac{7}{9} \approx 0.77$, or $\delta - 1 = -\frac{16}{25} \Rightarrow \frac{9}{25} = 0.36$; thus $\delta = \frac{9}{25} = 0.36$.

14. Step 1: $\left|x - \frac{1}{2}\right| < \delta \Rightarrow -\delta < x - \frac{1}{2} < \delta \Rightarrow -\delta + \frac{1}{2} < x < \delta + \frac{1}{2}$
Step 2: From the graph, $-\delta + \frac{1}{2} = \frac{1}{2.01} \Rightarrow \delta = \frac{1}{2} - \frac{1}{2.01} \approx 0.00248$, or $\delta + \frac{1}{2} = \frac{1}{1.99} \Rightarrow \delta = \frac{1}{1.99} - \frac{1}{2} \approx 0.00251$; thus $\delta = 0.00248$.

15. Step 1: $|(x + 1) - 5| < 0.01 \Rightarrow |x - 4| < 0.01 \Rightarrow -0.01 < x - 4 < 0.01 \Rightarrow 3.99 < x < 4.01$
Step 2: $|x - 4| < \delta \Rightarrow -\delta < x - 4 < \delta \Rightarrow -\delta + 4 < x < \delta + 4 \Rightarrow \delta = 0.01$.

16. Step 1: $|(2x - 2) - (-6)| < 0.02 \Rightarrow |2x + 4| < 0.02 \Rightarrow -0.02 < 2x + 4 < 0.02 \Rightarrow -4.02 < 2x < -3.98$ $\Rightarrow -2.01 < x < -1.99$
Step 2: $|x - (-2)| < \delta \Rightarrow -\delta < x + 2 < \delta \Rightarrow -\delta - 2 < x < \delta - 2 \Rightarrow \delta = 0.01$.

17. Step 1: $\left|\sqrt{x + 1} - 1\right| < 0.1 \Rightarrow -0.1 < \sqrt{x + 1} - 1 < 0.1 \Rightarrow 0.9 < \sqrt{x + 1} < 1.1 \Rightarrow 0.81 < x + 1 < 1.21$ $\Rightarrow -0.19 < x < 0.21$
Step 2: $|x - 0| < \delta \Rightarrow -\delta < x < \delta$. Then, $-\delta = -0.19 \Rightarrow \delta = 0.19$ or $\delta = 0.21$; thus, $\delta = 0.19$.

18. Step 1: $\left|\sqrt{x}-\frac{1}{2}\right|<0.1 \Rightarrow -0.1<\sqrt{x}-\frac{1}{2}<0.1 \Rightarrow 0.4<\sqrt{x}<0.6 \Rightarrow 0.16<x<0.36$
Step 2: $\left|x-\frac{1}{4}\right|<\delta \Rightarrow -\delta<x-\frac{1}{4}<\delta \Rightarrow -\delta+\frac{1}{4}<x<\delta+\frac{1}{4}$.
Then, $-\delta+\frac{1}{4}=0.16 \Rightarrow \delta=0.09$ or $\delta+\frac{1}{4}=0.36 \Rightarrow \delta=0.11$; thus $\delta=0.09$.

19. Step 1: $\left|\sqrt{19-x}-3\right|<1 \Rightarrow -1<\sqrt{19-x}-3<1 \Rightarrow 2<\sqrt{19-x}<4 \Rightarrow 4<19-x<16$
$\Rightarrow -4>x-19>-16 \Rightarrow 15>x>3$ or $3<x<15$
Step 2: $|x-10|<\delta \Rightarrow -\delta<x-10<\delta \Rightarrow -\delta+10<x<\delta+10$.
Then $-\delta+10=3 \Rightarrow \delta=7$, or $\delta+10=15 \Rightarrow \delta=5$; thus $\delta=5$.

20. Step 1: $\left|\sqrt{x-7}-4\right|<1 \Rightarrow -1<\sqrt{x-7}-4<1 \Rightarrow 3<\sqrt{x-7}<5 \Rightarrow 9<x-7<25 \Rightarrow 16<x<32$
Step 2: $|x-23|<\delta \Rightarrow -\delta<x-23<\delta \Rightarrow -\delta+23<x<\delta+23$.
Then $-\delta+23=16 \Rightarrow \delta=7$, or $\delta+23=32 \Rightarrow \delta=9$; thus $\delta=7$.

21. Step 1: $\left|\frac{1}{x}-\frac{1}{4}\right|<0.05 \Rightarrow -0.05<\frac{1}{x}-\frac{1}{4}<0.05 \Rightarrow 0.2<\frac{1}{x}<0.3 \Rightarrow \frac{10}{2}>x>\frac{10}{3}$ or $\frac{10}{3}<x<5$.
Step 2: $|x-4|<\delta \Rightarrow -\delta<x-4<\delta \Rightarrow -\delta+4<x<\delta+4$.
Then $-\delta+4=\frac{10}{3}$ or $\delta=\frac{2}{3}$, or $\delta+4=5$ or $\delta=1$; thus $\delta=\frac{2}{3}$.

22. Step 1: $|x^2-3|<0.1 \Rightarrow -0.1<x^2-3<0.1 \Rightarrow 2.9<x^2<3.1 \Rightarrow \sqrt{2.9}<x<\sqrt{3.1}$
Step 2: $\left|x-\sqrt{3}\right|<\delta \Rightarrow -\delta<x-\sqrt{3}<\delta \Rightarrow -\delta+\sqrt{3}<x<\delta+\sqrt{3}$.
Then $-\delta+\sqrt{3}=\sqrt{2.9} \Rightarrow \delta=\sqrt{3}-\sqrt{2.9}\approx 0.0291$, or $\delta+\sqrt{3}=\sqrt{3.1} \Rightarrow \delta=\sqrt{3.1}-\sqrt{3}\approx 0.0286$; thus $\delta=0.0286$.

23. Step 1: $|x^2-4|<0.5 \Rightarrow -0.5<x^2-4<0.5 \Rightarrow 3.5<x^2<4.5 \Rightarrow \sqrt{3.5}<|x|<\sqrt{4.5} \Rightarrow -\sqrt{4.5}<x<-\sqrt{3.5}$, for x near -2.
Step 2: $|x-(-2)|<\delta \Rightarrow -\delta<x+2<\delta \Rightarrow -\delta-2<x<\delta-2$.
Then $-\delta-2=-\sqrt{4.5} \Rightarrow \delta=\sqrt{4.5}-2\approx 0.1213$, or $\delta-2=-\sqrt{3.5} \Rightarrow \delta=2-\sqrt{3.5}\approx 0.1292$; thus $\delta=\sqrt{4.5}-2\approx 0.12$.

24. Step 1: $\left|\frac{1}{x}-(-1)\right|<0.1 \Rightarrow -0.1<\frac{1}{x}+1<0.1 \Rightarrow -\frac{11}{10}<\frac{1}{x}<-\frac{9}{10} \Rightarrow -\frac{10}{11}>x>-\frac{10}{9}$ or $-\frac{10}{9}<x<-\frac{10}{11}$.
Step 2: $|x-(-1)|<\delta \Rightarrow -\delta<x+1<\delta \Rightarrow -\delta-1<x<\delta-1$.
Then $-\delta-1=-\frac{10}{9} \Rightarrow \delta=\frac{1}{9}$, or $\delta-1=-\frac{10}{11} \Rightarrow \delta=\frac{1}{11}$; thus $\delta=\frac{1}{11}$.

25. Step 1: $|(x^2-5)-11|<1 \Rightarrow |x^2-16|<1 \Rightarrow -1<x^2-16<1 \Rightarrow 15<x^2<17 \Rightarrow \sqrt{15}<x<\sqrt{17}$.
Step 2: $|x-4|<\delta \Rightarrow -\delta<x-4<\delta \Rightarrow -\delta+4<x<\delta+4$.
Then $-\delta+4=\sqrt{15} \Rightarrow \delta=4-\sqrt{15}\approx 0.1270$, or $\delta+4=\sqrt{17} \Rightarrow \delta=\sqrt{17}-4\approx 0.1231$; thus $\delta=\sqrt{17}-4\approx 0.12$.

26. Step 1: $\left|\frac{120}{x}-5\right|<1 \Rightarrow -1<\frac{120}{x}-5<1 \Rightarrow 4<\frac{120}{x}<6 \Rightarrow \frac{1}{4}>\frac{x}{120}>\frac{1}{6} \Rightarrow 30>x>20$ or $20<x<30$.
Step 2: $|x-24|<\delta \Rightarrow -\delta<x-24<\delta \Rightarrow -\delta+24<x<\delta+24$.
Then $-\delta+24=20 \Rightarrow \delta=4$, or $\delta+24=30 \Rightarrow \delta=6$; thus $\Rightarrow \delta=4$.

27. Step 1: $|mx-2m|<0.03 \Rightarrow -0.03<mx-2m<0.03 \Rightarrow -0.03+2m<mx<0.03+2m \Rightarrow$
$2-\frac{0.03}{m}<x<2+\frac{0.03}{m}$.
Step 2: $|x-2|<\delta \Rightarrow -\delta<x-2<\delta \Rightarrow -\delta+2<x<\delta+2$.
Then $-\delta+2=2-\frac{0.03}{m} \Rightarrow \delta=\frac{0.03}{m}$, or $\delta+2=2+\frac{0.03}{m} \Rightarrow \delta=\frac{0.03}{m}$. In either case, $\delta=\frac{0.03}{m}$.

28. Step 1: $|mx - 3m| < c \Rightarrow -c < mx - 3m < c \Rightarrow -c + 3m < mx < c + 3m \Rightarrow 3 - \frac{c}{m} < x < 3 + \frac{c}{m}$
Step 2: $|x - 3| < \delta \Rightarrow -\delta < x - 3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3$.
Then $-\delta + 3 = 3 - \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$, or $\delta + 3 = 3 + \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$. In either case, $\delta = \frac{c}{m}$.

29. Step 1: $\left|(mx + b) - \left(\frac{m}{2} + b\right)\right| < c \Rightarrow -c < mx - \frac{m}{2} < c \Rightarrow -c + \frac{m}{2} < mx < c + \frac{m}{2} \Rightarrow \frac{1}{2} - \frac{c}{m} < x < \frac{1}{2} + \frac{c}{m}$.
Step 2: $\left|x - \frac{1}{2}\right| < \delta \Rightarrow -\delta < x - \frac{1}{2} < \delta \Rightarrow -\delta + \frac{1}{2} < x < \delta + \frac{1}{2}$.
Then $-\delta + \frac{1}{2} = \frac{1}{2} - \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$, or $\delta + \frac{1}{2} = \frac{1}{2} + \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$. In either case, $\delta = \frac{c}{m}$.

30. Step 1: $|(mx + b) - (m + b)| < 0.05 \Rightarrow -0.05 < mx - m < 0.05 \Rightarrow -0.05 + m < mx < 0.05 + m$
$\Rightarrow 1 - \frac{0.05}{m} < x < 1 + \frac{0.05}{m}$.
Step 2: $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow -\delta + 1 < x < \delta + 1$.
Then $-\delta + 1 = 1 - \frac{0.05}{m} \Rightarrow \delta = \frac{0.05}{m}$, or $\delta + 1 = 1 + \frac{0.05}{m} \Rightarrow \delta = \frac{0.05}{m}$. In either case, $\delta = \frac{0.05}{m}$.

31. $\lim_{x \to 3} (3 - 2x) = 3 - 2(3) = -3$
Step 1: $|(3 - 2x) - (-3)| < 0.02 \Rightarrow -0.02 < 6 - 2x < 0.02 \Rightarrow -6.02 < -2x < -5.98 \Rightarrow 3.01 > x > 2.99$ or $2.99 < x < 3.01$.
Step 2: $0 < |x - 3| < \delta \Rightarrow -\delta < x - 3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3$.
Then $-\delta + 3 = 2.99 \Rightarrow \delta = 0.01$, or $\delta + 3 = 3.01 \Rightarrow \delta = 0.01$; thus $\delta = 0.01$.

32. $\lim_{x \to -1} (-3x - 2) = (-3)(-1) - 2 = 1$
Step 1: $|(-3x - 2) - 1| < 0.03 \Rightarrow -0.03 < -3x - 3 < 0.03 \Rightarrow 0.01 > x + 1 > -0.01 \Rightarrow -1.01 < x < -0.99$.
Step 2: $|x - (-1)| < \delta \Rightarrow -\delta < x + 1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$.
Then $-\delta - 1 = -1.01 \Rightarrow \delta = 0.01$, or $\delta - 1 = -0.99 \Rightarrow \delta = 0.01$; thus $\delta = 0.01$.

33. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{(x - 2)} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4, x \neq 2$
Step 1: $\left|\left(\frac{x^2 - 4}{x - 2}\right) - 4\right| < 0.05 \Rightarrow -0.05 < \frac{(x + 2)(x - 2)}{(x - 2)} - 4 < 0.05 \Rightarrow 3.95 < x + 2 < 4.05, x \neq 2$
$\Rightarrow 1.95 < x < 2.05, x \neq 2$.
Step 2: $|x - 2| < \delta \Rightarrow -\delta < x - 2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2$.
Then $-\delta + 2 = 1.95 \Rightarrow \delta = 0.05$, or $\delta + 2 = 2.05 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

34. $\lim_{x \to -5} \frac{x^2 + 6x + 5}{x + 5} = \lim_{x \to -5} \frac{(x + 5)(x + 1)}{(x + 5)} = \lim_{x \to -5} (x + 1) = -4, x \neq -5$.
Step 1: $\left|\left(\frac{x^2 + 6x + 5}{x + 5}\right) - (-4)\right| < 0.05 \Rightarrow -0.05 < \frac{(x + 5)(x + 1)}{(x + 5)} + 4 < 0.05 \Rightarrow -4.05 < x + 1 < -3.95, x \neq -5$
$\Rightarrow -5.05 < x < -4.95, x \neq -5$.
Step 2: $|x - (-5)| < \delta \Rightarrow -\delta < x + 5 < \delta \Rightarrow -\delta - 5 < x < \delta - 5$.
Then $-\delta - 5 = -5.05 \Rightarrow \delta = 0.05$, or $\delta - 5 = -4.95 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

35. $\lim_{x \to -3} \sqrt{1 - 5x} = \sqrt{1 - 5(-3)} = \sqrt{16} = 4$
Step 1: $\left|\sqrt{1 - 5x} - 4\right| < 0.5 \Rightarrow -0.5 < \sqrt{1 - 5x} - 4 < 0.5 \Rightarrow 3.5 < \sqrt{1 - 5x} < 4.5 \Rightarrow 12.25 < 1 - 5x < 20.25$
$\Rightarrow 11.25 < -5x < 19.25 \Rightarrow -3.85 < x < -2.25$.
Step 2: $|x - (-3)| < \delta \Rightarrow -\delta < x + 3 < \delta \Rightarrow -\delta - 3 < x < \delta - 3$.
Then $-\delta - 3 = -3.85 \Rightarrow \delta = 0.85$, or $\delta - 3 = -2.25 \Rightarrow 0.75$; thus $\delta = 0.75$.

36. $\lim_{x \to 2} \frac{4}{x} = \frac{4}{2} = 2$
Step 1: $\left|\frac{4}{x} - 2\right| < 0.4 \Rightarrow -0.4 < \frac{4}{x} - 2 < 0.4 \Rightarrow 1.6 < \frac{4}{x} < 2.4 \Rightarrow \frac{10}{16} > \frac{x}{4} > \frac{10}{24} \Rightarrow \frac{10}{4} > x > \frac{10}{6}$ or $\frac{5}{3} < x < \frac{5}{2}$.

Step 2: $|x-2|<\delta \Rightarrow -\delta<x-2<\delta \Rightarrow -\delta+2<x<\delta+2$.
Then $-\delta+2=\frac{5}{3} \Rightarrow \delta=\frac{1}{3}$, or $\delta+2=\frac{5}{2} \Rightarrow \delta=\frac{1}{2}$; thus $\delta=\frac{1}{3}$.

37. Step 1: $|(9-x)-5|<\epsilon \Rightarrow -\epsilon<4-x<\epsilon \Rightarrow -\epsilon-4<-x<\epsilon-4 \Rightarrow \epsilon+4>x>4-\epsilon \Rightarrow 4-\epsilon<x<4+\epsilon$.
Step 2: $|x-4|<\delta \Rightarrow -\delta<x-4<\delta \Rightarrow -\delta+4<x<\delta+4$.
Then $-\delta+4=-\epsilon+4 \Rightarrow \delta=\epsilon$, or $\delta+4=\epsilon+4 \Rightarrow \delta=\epsilon$. Thus choose $\delta=\epsilon$.

38. Step 1: $|(3x-7)-2|<\epsilon \Rightarrow -\epsilon<3x-9<\epsilon \Rightarrow 9-\epsilon<3x<9+\epsilon \Rightarrow 3-\frac{\epsilon}{3}<x<3+\frac{\epsilon}{3}$.
Step 2: $|x-3|<\delta \Rightarrow -\delta<x-3<\delta \Rightarrow -\delta+3<x<\delta+3$.
Then $-\delta+3=3-\frac{\epsilon}{3} \Rightarrow \delta=\frac{\epsilon}{3}$, or $\delta+3=3+\frac{\epsilon}{3} \Rightarrow \delta=\frac{\epsilon}{3}$. Thus choose $\delta=\frac{\epsilon}{3}$.

39. Step 1: $\left|\sqrt{x-5}-2\right|<\epsilon \Rightarrow -\epsilon<\sqrt{x-5}-2<\epsilon \Rightarrow 2-\epsilon<\sqrt{x-5}<2+\epsilon \Rightarrow (2-\epsilon)^2<x-5<(2+\epsilon)^2$
$\Rightarrow (2-\epsilon)^2+5<x<(2+\epsilon)^2+5$.
Step 2: $|x-9|<\delta \Rightarrow -\delta<x-9<\delta \Rightarrow -\delta+9<x<\delta+9$.
Then $-\delta+9=\epsilon^2-4\epsilon+9 \Rightarrow \delta=4\epsilon-\epsilon^2$, or $\delta+9=\epsilon^2+4\epsilon+9 \Rightarrow \delta=4\epsilon+\epsilon^2$. Thus choose the smaller distance, $\delta=4\epsilon-\epsilon^2$.

40. Step 1: $\left|\sqrt{4-x}-2\right|<\epsilon \Rightarrow -\epsilon<\sqrt{4-x}-2<\epsilon \Rightarrow 2-\epsilon<\sqrt{4-x}<2+\epsilon \Rightarrow (2-\epsilon)^2<4-x<(2+\epsilon)^2$
$\Rightarrow -(2+\epsilon)^2<x-4<-(2-\epsilon)^2 \Rightarrow -(2+\epsilon)^2+4<x<-(2-\epsilon)^2+4$.
Step 2: $|x-0|<\delta \Rightarrow -\delta<x<\delta$.
Then $-\delta=-(2+\epsilon)^2+4=-\epsilon^2-4\epsilon \Rightarrow \delta=4\epsilon+\epsilon^2$, or $\delta=-(2-\epsilon)^2+4=4\epsilon-\epsilon^2$. Thus choose the smaller distance, $\delta=4\epsilon-\epsilon^2$.

41. Step 1: For $x\neq 1$, $|x^2-1|<\epsilon \Rightarrow -\epsilon<x^2-1<\epsilon \Rightarrow 1-\epsilon<x^2<1+\epsilon \Rightarrow \sqrt{1-\epsilon}<|x|<\sqrt{1+\epsilon}$
$\Rightarrow \sqrt{1-\epsilon}<x<\sqrt{1+\epsilon}$ near $x=1$.
Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow -\delta+1<x<\delta+1$.
Then $-\delta+1=\sqrt{1-\epsilon} \Rightarrow \delta=1-\sqrt{1-\epsilon}$, or $\delta+1=\sqrt{1+\epsilon} \Rightarrow \delta=\sqrt{1+\epsilon}-1$. Choose $\delta=\min\left\{1-\sqrt{1-\epsilon}, \sqrt{1+\epsilon}-1\right\}$, that is, the smaller of the two distances.

42. Step 1: For $x\neq -2$, $|x^2-4|<\epsilon \Rightarrow -\epsilon<x^2-4<\epsilon \Rightarrow 4-\epsilon<x^2<4+\epsilon \Rightarrow \sqrt{4-\epsilon}<|x|<\sqrt{4+\epsilon}$
$\Rightarrow -\sqrt{4+\epsilon}<x<-\sqrt{4-\epsilon}$ near $x=-2$.
Step 2: $|x-(-2)|<\delta \Rightarrow -\delta<x+2<\delta \Rightarrow -\delta-2<x<\delta-2$.
Then $-\delta-2=-\sqrt{4+\epsilon} \Rightarrow \delta=\sqrt{4+\epsilon}-2$, or $\delta-2=-\sqrt{4-\epsilon} \Rightarrow \delta=2-\sqrt{4-\epsilon}$. Choose $\delta=\min\left\{\sqrt{4+\epsilon}-2, 2-\sqrt{4-\epsilon}\right\}$.

43. Step 1: $\left|\frac{1}{x}-1\right|<\epsilon \Rightarrow -\epsilon<\frac{1}{x}-1<\epsilon \Rightarrow 1-\epsilon<\frac{1}{x}<1+\epsilon \Rightarrow \frac{1}{1+\epsilon}<x<\frac{1}{1-\epsilon}$.
Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow 1-\delta<x<1+\delta$.
Then $1-\delta=\frac{1}{1+\epsilon} \Rightarrow \delta=1-\frac{1}{1+\epsilon}=\frac{\epsilon}{1+\epsilon}$, or $1+\delta=\frac{1}{1-\epsilon} \Rightarrow \delta=\frac{1}{1-\epsilon}-1=\frac{\epsilon}{1-\epsilon}$.
Choose $\delta=\frac{\epsilon}{1+\epsilon}$, the smaller of the two distances.

44. Step 1: $\left|\frac{1}{x^2}-\frac{1}{3}\right|<\epsilon \Rightarrow -\epsilon<\frac{1}{x^2}-\frac{1}{3}<\epsilon \Rightarrow \frac{1}{3}-\epsilon<\frac{1}{x^2}<\frac{1}{3}+\epsilon \Rightarrow \frac{1-3\epsilon}{3}<\frac{1}{x^2}<\frac{1+3\epsilon}{3} \Rightarrow \frac{3}{1-3\epsilon}>x^2>\frac{3}{1+3\epsilon}$
$\Rightarrow \sqrt{\frac{3}{1+3\epsilon}}<|x|<\sqrt{\frac{3}{1-3\epsilon}}$, or $\sqrt{\frac{3}{1+3\epsilon}}<x<\sqrt{\frac{3}{1-3\epsilon}}$ for x near $\sqrt{3}$.

Step 2: $\left|x - \sqrt{3}\right| < \delta \Rightarrow -\delta < x - \sqrt{3} < \delta \Rightarrow \sqrt{3} - \delta < x < \sqrt{3} + \delta.$

Then $\sqrt{3} - \delta = \sqrt{\frac{3}{1+3\epsilon}} \Rightarrow \delta = \sqrt{3} - \sqrt{\frac{3}{1+3\epsilon}}$, or $\sqrt{3} + \delta = \sqrt{\frac{3}{1-3\epsilon}} \Rightarrow \delta = \sqrt{\frac{3}{1-3\epsilon}} - \sqrt{3}.$

Choose $\delta = \min\left\{\sqrt{3} - \sqrt{\frac{3}{1+3\epsilon}}, \sqrt{\frac{3}{1-3\epsilon}} - \sqrt{3}\right\}.$

45. Step 1: $\left|\left(\frac{x^2-9}{x+3}\right) - (-6)\right| < \epsilon \Rightarrow -\epsilon < (x-3) + 6 < \epsilon, x \neq -3 \Rightarrow -\epsilon < x + 3 < \epsilon \Rightarrow -\epsilon - 3 < x < \epsilon - 3.$

Step 2: $|x - (-3)| < \delta \Rightarrow -\delta < x + 3 < \delta \Rightarrow -\delta - 3 < x < \delta - 3.$

Then $-\delta - 3 = -\epsilon - 3 \Rightarrow \delta = \epsilon$, or $\delta - 3 = \epsilon - 3 \Rightarrow \delta = \epsilon$. Choose $\delta = \epsilon$.

46. Step 1: $\left|\left(\frac{x^2-1}{x-1}\right) - 2\right| < \epsilon \Rightarrow -\epsilon < (x+1) - 2 < \epsilon, x \neq 1 \Rightarrow 1 - \epsilon < x < 1 + \epsilon.$

Step 2: $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow 1 - \delta < x < 1 + \delta.$

Then $1 - \delta = 1 - \epsilon \Rightarrow \delta = \epsilon$, or $1 + \delta = 1 + \epsilon \Rightarrow \delta = \epsilon$. Choose $\delta = \epsilon$.

47. Step 1: $x < 1$: $|(4 - 2x) - 2| < \epsilon \Rightarrow 0 < 2 - 2x < \epsilon$ since $x < 1$. Thus, $1 - \frac{\epsilon}{2} < x < 0$;

$x \geq 1$: $|(6x - 4) - 2| < \epsilon \Rightarrow 0 \leq 6x - 6 < \epsilon$ since $x \geq 1$. Thus, $1 \leq x < 1 + \frac{\epsilon}{6}$.

Step 2: $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow 1 - \delta < x < 1 + \delta.$

Then $1 - \delta = 1 - \frac{\epsilon}{2} \Rightarrow \delta = \frac{\epsilon}{2}$, or $1 + \delta = 1 + \frac{\epsilon}{6} \Rightarrow \delta = \frac{\epsilon}{6}$. Choose $\delta = \frac{\epsilon}{6}$.

48. Step 1: $x < 0$: $|2x - 0| < \epsilon \Rightarrow -\epsilon < 2x < 0 \Rightarrow -\frac{\epsilon}{2} < x < 0$;

$x \geq 0$: $\left|\frac{x}{2} - 0\right| < \epsilon \Rightarrow 0 \leq x < 2\epsilon$.

Step 2: $|x - 0| < \delta \Rightarrow -\delta < x < \delta.$

Then $-\delta = -\frac{\epsilon}{2} \Rightarrow \delta = \frac{\epsilon}{2}$, or $\delta = 2\epsilon \Rightarrow \delta = 2\epsilon$. Choose $\delta = \frac{\epsilon}{2}$.

49. By the figure, $-x \leq x \sin \frac{1}{x} \leq x$ for all $x > 0$ and $-x \geq x \sin \frac{1}{x} \geq x$ for $x < 0$. Since $\lim_{x \to 0} (-x) = \lim_{x \to 0} x = 0$, then by the sandwich theorem, in either case, $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

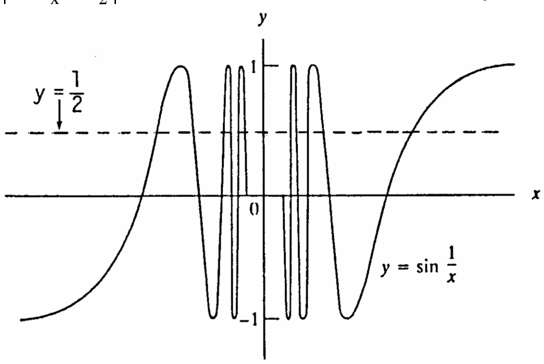
50. By the figure, $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$ for all x except possibly at $x = 0$. Since $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$, then by the sandwich theorem, $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

51. As x approaches the value 0, the values of $g(x)$ approach k. Thus for every number $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - 0| < \delta \Rightarrow |g(x) - k| < \epsilon$.

52. Write $x = h + c$. Then $0 < |x - c| < \delta \Leftrightarrow -\delta < x - c < \delta, x \neq c \Leftrightarrow -\delta < (h + c) - c < \delta, h + c \neq c$
$\Leftrightarrow -\delta < h < \delta, h \neq 0 \Leftrightarrow 0 < |h - 0| < \delta$.
Thus, $\lim_{x \to c} f(x) = L \Leftrightarrow$ for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$
$\Leftrightarrow |f(h + c) - L| < \epsilon$ whenever $0 < |h - 0| < \delta \Leftrightarrow \lim_{h \to 0} f(h + c) = L$.

53. Let $f(x) = x^2$. The function values do get closer to -1 as x approaches 0, but $\lim_{x \to 0} f(x) = 0$, not -1. The function $f(x) = x^2$ never gets <u>arbitrarily</u> <u>close</u> to -1 for x near 0.

54. Let $f(x) = \sin x$, $L = \frac{1}{2}$, and $x_0 = 0$. There exists a value of x (namely, $x = \frac{\pi}{6}$) for which $\left|\sin x - \frac{1}{2}\right| < \epsilon$ for any given $\epsilon > 0$. However, $\lim_{x \to 0} \sin x = 0$, not $\frac{1}{2}$. The wrong statement does not require x to be arbitrarily close to x_0. As another\ example, let $g(x) = \sin \frac{1}{x}$, $L = \frac{1}{2}$, and $x_0 = 0$. We can choose infinitely many values of x near 0 such that $\sin \frac{1}{x} = \frac{1}{2}$ as you can see from the accompanying figure. However, $\lim_{x \to 0} \sin \frac{1}{x}$ fails to exist. The wrong statement does not require <u>all</u> values of x arbitrarily close to $x_0 = 0$ to lie within $\epsilon > 0$ of $L = \frac{1}{2}$. Again you can see from the figure that there are also infinitely many values of x near 0 such that $\sin \frac{1}{x} = 0$. If we choose $\epsilon < \frac{1}{4}$ we cannot satisfy the inequality $\left|\sin \frac{1}{x} - \frac{1}{2}\right| < \epsilon$ for all values of x sufficiently near $x_0 = 0$.



55. $|A - 9| \le 0.01 \Rightarrow -0.01 \le \pi\left(\frac{x}{2}\right)^2 - 9 \le 0.01 \Rightarrow 8.99 \le \frac{\pi x^2}{4} \le 9.01 \Rightarrow \frac{4}{\pi}(8.99) \le x^2 \le \frac{4}{\pi}(9.01)$ $\Rightarrow 2\sqrt{\frac{8.99}{\pi}} \le x \le 2\sqrt{\frac{9.01}{\pi}}$ or $3.384 \le x \le 3.387$. To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

56. $V = RI \Rightarrow \frac{V}{R} = I \Rightarrow \left|\frac{V}{R} - 5\right| \le 0.1 \Rightarrow -0.1 \le \frac{120}{R} - 5 \le 0.1 \Rightarrow 4.9 \le \frac{120}{R} \le 5.1 \Rightarrow \frac{10}{49} \ge \frac{R}{120} \ge \frac{10}{51} \Rightarrow$ $\frac{(120)(10)}{51} \le R \le \frac{(120)(10)}{49} \Rightarrow 23.53 \le R \le 24.48$.
To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

57. (a) $-\delta < x - 1 < 0 \Rightarrow 1 - \delta < x < 1 \Rightarrow f(x) = x$. Then $|f(x) - 2| = |x - 2| = 2 - x > 2 - 1 = 1$. That is, $|f(x) - 2| \ge 1 \ge \frac{1}{2}$ no matter how small δ is taken when $1 - \delta < x < 1 \Rightarrow \lim_{x \to 1} f(x) \ne 2$.
(b) $0 < x - 1 < \delta \Rightarrow 1 < x < 1 + \delta \Rightarrow f(x) = x + 1$. Then $|f(x) - 1| = |(x + 1) - 1| = |x| = x > 1$. That is, $|f(x) - 1| \ge 1$ no matter how small δ is taken when $1 < x < 1 + \delta \Rightarrow \lim_{x \to 1} f(x) \ne 1$.
(c) $-\delta < x - 1 < 0 \Rightarrow 1 - \delta < x < 1 \Rightarrow f(x) = x$. Then $|f(x) - 1.5| = |x - 1.5| = 1.5 - x > 1.5 - 1 = 0.5$. Also, $0 < x - 1 < \delta \Rightarrow 1 < x < 1 + \delta \Rightarrow f(x) = x + 1$. Then $|f(x) - 1.5| = |(x + 1) - 1.5| = |x - 0.5|$ $= x - 0.5 > 1 - 0.5 = 0.5$. Thus, no matter how small δ is taken, there exists a value of x such that $-\delta < x - 1 < \delta$ but $|f(x) - 1.5| \ge \frac{1}{2} \Rightarrow \lim_{x \to 1} f(x) \ne 1.5$.

58. (a) For $2 < x < 2 + \delta \Rightarrow h(x) = 2 \Rightarrow |h(x) - 4| = 2$. Thus for $\epsilon < 2$, $|h(x) - 4| \ge \epsilon$ whenever $2 < x < 2 + \delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \to 2} h(x) \ne 4$.
(b) For $2 < x < 2 + \delta \Rightarrow h(x) = 2 \Rightarrow |h(x) - 3| = 1$. Thus for $\epsilon < 1$, $|h(x) - 3| \ge \epsilon$ whenever $2 < x < 2 + \delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \to 2} h(x) \ne 3$.
(c) For $2 - \delta < x < 2 \Rightarrow h(x) = x^2$ so $|h(x) - 2| = |x^2 - 2|$. No matter how small $\delta > 0$ is chosen, x^2 is close to 4 when x is near 2 and to the left on the real line $\Rightarrow |x^2 - 2|$ will be close to 2. Thus if $\epsilon < 1$, $|h(x) - 2| \ge \epsilon$ whenever $2 - \delta < x < 2$ no mater how small we choose $\delta > 0 \Rightarrow \lim_{x \to 2} h(x) \ne 2$.

59. (a) For $3 - \delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x) - 4| \geq 0.8$. Thus for $\epsilon < 0.8$, $|f(x) - 4| \geq \epsilon$ whenever $3 - \delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \to 3} f(x) \neq 4$.

(b) For $3 < x < 3 + \delta \Rightarrow f(x) < 3 \Rightarrow |f(x) - 4.8| \geq 1.8$. Thus for $\epsilon < 1.8$, $|f(x) - 4.8| \geq \epsilon$ whenever $3 < x < 3 + \delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \to 3} f(x) \neq 4.8$.

(c) For $3 - \delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x) - 3| \geq 1.8$. Again, for $\epsilon < 1.8$, $|f(x) - 3| \geq \epsilon$ whenever $3 - \delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \to 3} f(x) \neq 3$.

60. (a) No matter how small we choose $\delta > 0$, for x near -1 satisfying $-1 - \delta < x < -1 + \delta$, the values of g(x) are near 1 $\Rightarrow |g(x) - 2|$ is near 1. Then, for $\epsilon = \frac{1}{2}$ we have $|g(x) - 2| \geq \frac{1}{2}$ for some x satisfying $-1 - \delta < x < -1 + \delta$, or $0 < |x + 1| < \delta \Rightarrow \lim_{x \to -1} g(x) \neq 2$.

(b) Yes, $\lim_{x \to -1} g(x) = 1$ because from the graph we can find a $\delta > 0$ such that $|g(x) - 1| < \epsilon$ if $0 < |x - (-1)| < \delta$.

61-66. Example CAS commands (values of del may vary for a specified eps):

Maple:

```
f := x -> (x^4-81)/(x-3);x0 := 3;
plot( f(x), x=x0-1..x0+1, color=black,                # (a)
      title="Section 2.3, #61(a)" );
L := limit( f(x), x=x0 );                              # (b)
epsilon := 0.2;                                        # (c)
plot( [f(x),L-epsilon,L+epsilon], x=x0-0.01..x0+0.01,
      color=black, linestyle=[1,3,3], title="Section 2.3, #61(c)" );
q := fsolve( abs( f(x)-L ) = epsilon, x=x0-1..x0+1 );  # (d)
delta := abs(x0-q);
plot( [f(x),L-epsilon,L+epsilon], x=x0-delta..x0+delta, color=black, title="Section 2.3, #61(d)" );
for eps in [0.1, 0.005, 0.001 ] do                     # (e)
  q := fsolve( abs( f(x)-L ) = eps, x=x0-1..x0+1 );
  delta := abs(x0-q);
  head := sprintf("Section 2.3, #61(e)\n epsilon = %5f, delta = %5f\n", eps, delta );
  print(plot( [f(x),L-eps,L+eps], x=x0-delta..x0+delta,
             color=black, linestyle=[1,3,3], title=head ));
end do:
```

Mathematica (assigned function and values for x0, eps and del may vary):

```
Clear[f, x]
y1: = L − eps; y2: = L + eps; x0 = 1;
f[x_]: = (3x^2 − (7x + 1)Sqrt[x] + 5)/(x − 1)
Plot[f[x], {x, x0 − 0.2, x0 + 0.2}]
L: = Limit[f[x], x → x0]
eps = 0.1; del = 0.2;
Plot[{f[x], y1, y2},{x, x0 − del, x0 + del}, PlotRange → {L − 2eps, L + 2eps}]
```

2.4 ONE-SIDED LIMITS

1. (a) True (b) True (c) False (d) True
   (e) True (f) True (g) False (h) False
   (i) False (j) False (k) True (l) False

2. (a) True (b) False (c) False (d) True
   (e) True (f) True (g) True (h) True
   (i) True (j) False (k) True

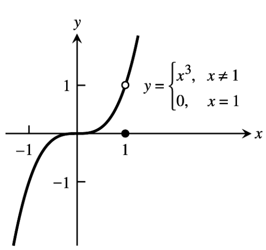
3. (a) $\lim_{x \to 2^+} f(x) = \frac{2}{2} + 1 = 2, \lim_{x \to 2^-} f(x) = 3 - 2 = 1$
   (b) No, $\lim_{x \to 2} f(x)$ does not exist because $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$
   (c) $\lim_{x \to 4^-} f(x) = \frac{4}{2} + 1 = 3, \lim_{x \to 4^+} f(x) = \frac{4}{2} + 1 = 3$
   (d) Yes, $\lim_{x \to 4} f(x) = 3$ because $3 = \lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x)$

4. (a) $\lim_{x \to 2^+} f(x) = \frac{2}{2} = 1, \lim_{x \to 2^-} f(x) = 3 - 2 = 1, f(2) = 2$
   (b) Yes, $\lim_{x \to 2} f(x) = 1$ because $1 = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$
   (c) $\lim_{x \to -1^-} f(x) = 3 - (-1) = 4, \lim_{x \to -1^+} f(x) = 3 - (-1) = 4$
   (d) Yes, $\lim_{x \to -1} f(x) = 4$ because $4 = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

5. (a) No, $\lim_{x \to 0^+} f(x)$ does not exist since $\sin\left(\frac{1}{x}\right)$ does not approach any single value as x approaches 0
   (b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$
   (c) $\lim_{x \to 0} f(x)$ does not exist because $\lim_{x \to 0^+} f(x)$ does not exist

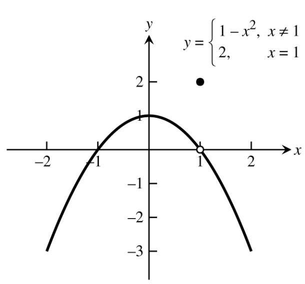
6. (a) Yes, $\lim_{x \to 0^+} g(x) = 0$ by the sandwich theorem since $-\sqrt{x} \leq g(x) \leq \sqrt{x}$ when $x > 0$
   (b) No, $\lim_{x \to 0^-} g(x)$ does not exist since $\sqrt{x}$ is not defined for $x < 0$
   (c) No, $\lim_{x \to 0} g(x)$ does not exist since $\lim_{x \to 0^-} g(x)$ does not exist

7. (a)



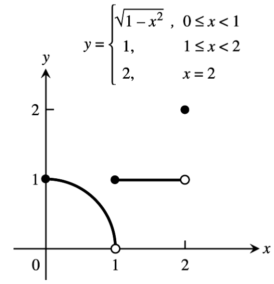
   (b) $\lim_{x \to 1^-} f(x) = 1 = \lim_{x \to 1^+} f(x)$
   (c) Yes, $\lim_{x \to 1} f(x) = 1$ since the right-hand and left-hand limits exist and equal 1

8. (a)

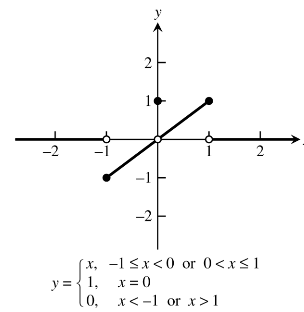


   (b) $\lim_{x \to 1^+} f(x) = 0 = \lim_{x \to 1^-} f(x)$
   (c) Yes, $\lim_{x \to 1} f(x) = 0$ since the right-hand and left-hand limits exist and equal 0

9. (a) domain: $0 \le x \le 2$
range: $0 < y \le 1$ and $y = 2$
(b) $\lim_{x \to c} f(x)$ exists for c belonging to $(0, 1) \cup (1, 2)$
(c) $x = 2$
(d) $x = 0$



10. (a) domain: $-\infty < x < \infty$
range: $-1 \le y \le 1$
(b) $\lim_{x \to c} f(x)$ exists for c belonging to $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
(c) none
(d) none



11. $\lim_{x \to -0.5^-} \sqrt{\frac{x+2}{x-1}} = \sqrt{\frac{-0.5+2}{-0.5+1}} = \sqrt{\frac{3/2}{1/2}} = \sqrt{3}$

12. $\lim_{x \to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}} = \sqrt{0} = 0$

13. $\lim_{x \to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right) = (2)\left(\frac{1}{2}\right) = 1$

14. $\lim_{x \to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right) = \left(\frac{1}{1+1}\right)\left(\frac{1+6}{1}\right)\left(\frac{3-1}{7}\right) = \left(\frac{1}{2}\right)\left(\frac{7}{1}\right)\left(\frac{2}{7}\right) = 1$

15. $$\lim_{h \to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h} = \lim_{h \to 0^+} \left(\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}\right)\left(\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}}\right)$$
$$= \lim_{h \to 0^+} \frac{(h^2+4h+5)-5}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} = \lim_{h \to 0^+} \frac{h(h+4)}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} = \frac{0+4}{\sqrt{5}+\sqrt{5}} = \frac{2}{\sqrt{5}}$$

16. $$\lim_{h \to 0^-} \frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h} = \lim_{h \to 0^-} \left(\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}\right)\left(\frac{\sqrt{6}+\sqrt{5h^2+11h+6}}{\sqrt{6}+\sqrt{5h^2+11h+6}}\right)$$
$$= \lim_{h \to 0^-} \frac{6-(5h^2+11h+6)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \lim_{h \to 0^-} \frac{-h(5h+11)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \frac{-(0+11)}{\sqrt{6}+\sqrt{6}} = -\frac{11}{2\sqrt{6}}$$

17. (a) $\lim_{x \to -2^+} (x+3)\frac{|x+2|}{x+2} = \lim_{x \to -2^+} (x+3)\frac{(x+2)}{(x+2)}$ $\quad (|x+2| = (x+2) \text{ for } x > -2)$
$= \lim_{x \to -2^+} (x+3) = ((-2)+3) = 1$

(b) $\lim_{x \to -2^-} (x+3)\frac{|x+2|}{x+2} = \lim_{x \to -2^-} (x+3)\left[\frac{-(x+2)}{(x+2)}\right]$ $\quad (|x+2| = -(x+2) \text{ for } x < -2)$
$= \lim_{x \to -2^-} (x+3)(-1) = -(-2+3) = -1$

18. (a) $\lim_{x \to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim_{x \to 1^+} \frac{\sqrt{2x}(x-1)}{(x-1)}$ $\quad (|x-1| = x-1 \text{ for } x > 1)$
$= \lim_{x \to 1^+} \sqrt{2x} = \sqrt{2}$

(b) $\lim_{x \to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim_{x \to 1^-} \frac{\sqrt{2x}(x-1)}{-(x-1)}$ $\quad (|x-1| = -(x-1) \text{ for } x < 1)$
$= \lim_{x \to 1^-} -\sqrt{2x} = -\sqrt{2}$

19. (a) $\lim_{\theta \to 3^+} \frac{\lfloor\theta\rfloor}{\theta} = \frac{3}{3} = 1$ (b) $\lim_{\theta \to 3^-} \frac{\lfloor\theta\rfloor}{\theta} = \frac{2}{3}$

20. (a) $\lim_{t \to 4^+} (t - \lfloor t \rfloor) = 4 - 4 = 0$ (b) $\lim_{t \to 4^-} (t - \lfloor t \rfloor) = 4 - 3 = 1$

21. $\lim_{\theta \to 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta} = \lim_{x \to 0} \frac{\sin x}{x} = 1$ (where $x = \sqrt{2}\theta$)

22. $\lim_{t \to 0} \frac{\sin kt}{t} = \lim_{t \to 0} \frac{k \sin kt}{kt} = \lim_{\theta \to 0} \frac{k \sin \theta}{\theta} = k \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = k \cdot 1 = k$ (where $\theta = kt$)

23. $\lim_{y \to 0} \frac{\sin 3y}{4y} = \frac{1}{4} \lim_{y \to 0} \frac{3 \sin 3y}{3y} = \frac{3}{4} \lim_{y \to 0} \frac{\sin 3y}{3y} = \frac{3}{4} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{3}{4}$ (where $\theta = 3y$)

24. $\lim_{h \to 0^-} \frac{h}{\sin 3h} = \lim_{h \to 0^-} \left(\frac{1}{3} \cdot \frac{3h}{\sin 3h}\right) = \frac{1}{3} \lim_{h \to 0^-} \frac{1}{\left(\frac{\sin 3h}{3h}\right)} = \frac{1}{3} \left(\frac{1}{\lim_{\theta \to 0^-} \frac{\sin \theta}{\theta}}\right) = \frac{1}{3} \cdot 1 = \frac{1}{3}$ (where $\theta = 3h$)

25. $\lim_{x \to 0} \frac{\tan 2x}{x} = \lim_{x \to 0} \frac{\left(\frac{\sin 2x}{\cos 2x}\right)}{x} = \lim_{x \to 0} \frac{\sin 2x}{x \cos 2x} = \left(\lim_{x \to 0} \frac{1}{\cos 2x}\right)\left(\lim_{x \to 0} \frac{2 \sin 2x}{2x}\right) = 1 \cdot 2 = 2$

26. $\lim_{t \to 0} \frac{2t}{\tan t} = 2 \lim_{t \to 0} \frac{t}{\left(\frac{\sin t}{\cos t}\right)} = 2 \lim_{t \to 0} \frac{t \cos t}{\sin t} = 2\left(\lim_{t \to 0} \cos t\right)\left(\frac{1}{\lim_{t \to 0} \frac{\sin t}{t}}\right) = 2 \cdot 1 \cdot 1 = 2$

27. $\lim_{x \to 0} \frac{x \csc 2x}{\cos 5x} = \lim_{x \to 0} \left(\frac{x}{\sin 2x} \cdot \frac{1}{\cos 5x}\right) = \left(\frac{1}{2} \lim_{x \to 0} \frac{2x}{\sin 2x}\right)\left(\lim_{x \to 0} \frac{1}{\cos 5x}\right) = \left(\frac{1}{2} \cdot 1\right)(1) = \frac{1}{2}$

28. $\lim_{x \to 0} 6x^2(\cot x)(\csc 2x) = \lim_{x \to 0} \frac{6x^2 \cos x}{\sin x \sin 2x} = \lim_{x \to 0} \left(3 \cos x \cdot \frac{x}{\sin x} \cdot \frac{2x}{\sin 2x}\right) = 3 \cdot 1 \cdot 1 = 3$

29. $\lim_{x \to 0} \frac{x + x \cos x}{\sin x \cos x} = \lim_{x \to 0} \left(\frac{x}{\sin x \cos x} + \frac{x \cos x}{\sin x \cos x}\right) = \lim_{x \to 0} \left(\frac{x}{\sin x} \cdot \frac{1}{\cos x}\right) + \lim_{x \to 0} \frac{x}{\sin x}$
$= \lim_{x \to 0} \left(\frac{1}{\frac{\sin x}{x}}\right) \cdot \lim_{x \to 0} \left(\frac{1}{\cos x}\right) + \lim_{x \to 0} \left(\frac{1}{\frac{\sin x}{x}}\right) = (1)(1) + 1 = 2$

30. $\lim_{x \to 0} \frac{x^2 - x + \sin x}{2x} = \lim_{x \to 0} \left(\frac{x}{2} - \frac{1}{2} + \frac{1}{2}\left(\frac{\sin x}{x}\right)\right) = 0 - \frac{1}{2} + \frac{1}{2}(1) = 0$

31. $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\sin 2\theta} = \lim_{\theta \to 0} \frac{(1 - \cos \theta)(1 + \cos \theta)}{(2 \sin \theta \cos \theta)(1 + \cos \theta)} = \lim_{\theta \to 0} \frac{1 - \cos^2 \theta}{(2 \sin \theta \cos \theta)(1 + \cos \theta)} = \lim_{\theta \to 0} \frac{\sin^2 \theta}{(2 \sin \theta \cos \theta)(1 + \cos \theta)}$
$= \lim_{\theta \to 0} \frac{\sin \theta}{(2 \cos \theta)(1 + \cos \theta)} = \frac{0}{(2)(2)} = 0$

32. $\lim_{x \to 0} \frac{x - x \cos x}{\sin^2 3x} = \lim_{x \to 0} \frac{x(1 - \cos x)}{\sin^2 3x} = \lim_{x \to 0} \frac{\frac{x(1 - \cos x)}{9x^2}}{\frac{\sin^2 3x}{9x^2}} = \lim_{x \to 0} \frac{\frac{1 - \cos x}{9x}}{\left(\frac{\sin 3x}{3x}\right)^2} = \frac{\frac{1}{9} \lim_{x \to 0} \left(\frac{1 - \cos x}{x}\right)}{\lim_{x \to 0} \left(\frac{\sin 3x}{3x}\right)^2} = \frac{\frac{1}{9}(0)}{1^2} = 0$

33. $\lim_{t \to 0} \frac{\sin(1 - \cos t)}{1 - \cos t} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ since $\theta = 1 - \cos t \to 0$ as $t \to 0$

34. $\lim_{h \to 0} \frac{\sin(\sin h)}{\sin h} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ since $\theta = \sin h \to 0$ as $h \to 0$

35. $\lim_{\theta \to 0} \frac{\sin \theta}{\sin 2\theta} = \lim_{\theta \to 0} \left(\frac{\sin \theta}{\sin 2\theta} \cdot \frac{2\theta}{2\theta}\right) = \frac{1}{2} \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{2\theta}{\sin 2\theta}\right) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$

36. $\lim_{x \to 0} \frac{\sin 5x}{\sin 4x} = \lim_{x \to 0} \left(\frac{\sin 5x}{\sin 4x} \cdot \frac{4x}{5x} \cdot \frac{5}{4}\right) = \frac{5}{4} \lim_{x \to 0} \left(\frac{\sin 5x}{5x} \cdot \frac{4x}{\sin 4x}\right) = \frac{5}{4} \cdot 1 \cdot 1 = \frac{5}{4}$

37. $\lim\limits_{\theta \to 0} \theta \cos\theta = 0 \cdot 1 = 0$

38. $\lim\limits_{\theta \to 0} \sin\theta \cot 2\theta = \lim\limits_{\theta \to 0} \sin\theta \frac{\cos 2\theta}{\sin 2\theta} = \lim\limits_{\theta \to 0} \sin\theta \frac{\cos 2\theta}{2\sin\theta\cos\theta} = \lim\limits_{\theta \to 0} \frac{\cos 2\theta}{2\cos\theta} = \frac{1}{2}$

39. $\lim\limits_{x \to 0} \frac{\tan 3x}{\sin 8x} = \lim\limits_{x \to 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x}\right) = \lim\limits_{x \to 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x} \cdot \frac{8x}{3x} \cdot \frac{3}{8}\right)$
$= \frac{3}{8} \lim\limits_{x \to 0} \left(\frac{1}{\cos 3x}\right)\left(\frac{\sin 3x}{3x}\right)\left(\frac{8x}{\sin 8x}\right) = \frac{3}{8} \cdot 1 \cdot 1 \cdot 1 = \frac{3}{8}$

40. $\lim\limits_{y \to 0} \frac{\sin 3y \cot 5y}{y \cot 4y} = \lim\limits_{y \to 0} \frac{\sin 3y \sin 4y \cos 5y}{y \cos 4y \sin 5y} = \lim\limits_{y \to 0} \left(\frac{\sin 3y}{y}\right)\left(\frac{\sin 4y}{\cos 4y}\right)\left(\frac{\cos 5y}{\sin 5y}\right)\left(\frac{3\cdot 4\cdot 5y}{3\cdot 4\cdot 5y}\right)$
$= \lim\limits_{y \to 0} \left(\frac{\sin 3y}{3y}\right)\left(\frac{\sin 4y}{4y}\right)\left(\frac{5y}{\sin 5y}\right)\left(\frac{\cos 5y}{\cos 4y}\right)\left(\frac{3\cdot 4}{5}\right) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot \frac{12}{5} = \frac{12}{5}$

41. $\lim\limits_{\theta \to 0} \frac{\tan\theta}{\theta^2 \cot 3\theta} = \lim\limits_{\theta \to 0} \frac{\frac{\sin\theta}{\cos\theta}}{\theta^2 \frac{\cos 3\theta}{\sin 3\theta}} = \lim\limits_{\theta \to 0} \frac{\sin\theta \sin 3\theta}{\theta^2 \cos\theta \cos 3\theta} = \lim\limits_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin 3\theta}{3\theta}\right)\left(\frac{3}{\cos\theta\cos 3\theta}\right) = (1)(1)\left(\frac{3}{1\cdot 1}\right) = 3$

42. $\lim\limits_{\theta \to 0} \frac{\theta \cot 4\theta}{\sin^2\theta \cot^2 2\theta} = \lim\limits_{\theta \to 0} \frac{\theta \frac{\cos 4\theta}{\sin 4\theta}}{\sin^2\theta \frac{\cos^2 2\theta}{\sin^2 2\theta}} = \lim\limits_{\theta \to 0} \frac{\theta \cos 4\theta \sin^2 2\theta}{\sin^2\theta \cos^2 2\theta \sin 4\theta} = \lim\limits_{\theta \to 0} \frac{\theta \cos 4\theta\,(2\sin\theta\cos\theta)^2}{\sin^2\theta\cos^2 2\theta \sin 4\theta} = \lim\limits_{\theta \to 0} \frac{\theta\cos 4\theta\,(4\sin^2\theta\cos^2\theta)}{\sin^2\theta\cos^2 2\theta\sin 4\theta}$
$= \lim\limits_{\theta \to 0} \frac{4\theta\cos 4\theta\cos^2\theta}{\cos^2 2\theta \sin 4\theta} = \lim\limits_{\theta \to 0} \left(\frac{4\theta}{\sin 4\theta}\right)\left(\frac{\cos 4\theta\cos^2\theta}{\cos^2 2\theta}\right) = \lim\limits_{\theta \to 0} \left(\frac{1}{\frac{\sin 4\theta}{4\theta}}\right)\left(\frac{\cos 4\theta\cos^2\theta}{\cos^2 2\theta}\right) = \left(\frac{1}{1}\right)\left(\frac{1\cdot 1^2}{1^2}\right) = 1$

43. Yes. If $\lim\limits_{x \to a^+} f(x) = L = \lim\limits_{x \to a^-} f(x)$, then $\lim\limits_{x \to a} f(x) = L$. If $\lim\limits_{x \to a^+} f(x) \neq \lim\limits_{x \to a^-} f(x)$, then $\lim\limits_{x \to a} f(x)$ does not exist.

44. Since $\lim\limits_{x \to c} f(x) = L$ if and only if $\lim\limits_{x \to c^+} f(x) = L$ and $\lim\limits_{x \to c^-} f(x) = L$, then $\lim\limits_{x \to c} f(x)$ can be found by calculating $\lim\limits_{x \to c^+} f(x)$.

45. If f is an odd function of x, then $f(-x) = -f(x)$. Given $\lim\limits_{x \to 0^+} f(x) = 3$, then $\lim\limits_{x \to 0^-} f(x) = -3$.

46. If f is an even function of x, then $f(-x) = f(x)$. Given $\lim\limits_{x \to 2^-} f(x) = 7$ then $\lim\limits_{x \to -2^+} f(x) = 7$. However, nothing can be said about $\lim\limits_{x \to -2^-} f(x)$ because we don't know $\lim\limits_{x \to 2^+} f(x)$.

47. $I = (5, 5+\delta) \Rightarrow 5 < x < 5+\delta$. Also, $\sqrt{x-5} < \epsilon \Rightarrow x - 5 < \epsilon^2 \Rightarrow x < 5 + \epsilon^2$. Choose $\delta = \epsilon^2$
$\Rightarrow \lim\limits_{x \to 5^+} \sqrt{x-5} = 0$.

48. $I = (4-\delta, 4) \Rightarrow 4 - \delta < x < 4$. Also, $\sqrt{4-x} < \epsilon \Rightarrow 4 - x < \epsilon^2 \Rightarrow x > 4 - \epsilon^2$. Choose $\delta = \epsilon^2$
$\Rightarrow \lim\limits_{x \to 4^-} \sqrt{4-x} = 0$.

49. As $x \to 0^-$ the number x is always negative. Thus, $\left|\frac{x}{|x|} - (-1)\right| < \epsilon \Rightarrow \left|\frac{x}{-x} + 1\right| < \epsilon \Rightarrow 0 < \epsilon$ which is always true independent of the value of x. Hence we can choose any $\delta > 0$ with $-\delta < x < 0 \Rightarrow \lim\limits_{x \to 0^-} \frac{x}{|x|} = -1$.

50. Since $x \to 2^+$ we have $x > 2$ and $|x - 2| = x - 2$. Then, $\left|\frac{x-2}{|x-2|} - 1\right| = \left|\frac{x-2}{x-2} - 1\right| < \epsilon \Rightarrow 0 < \epsilon$ which is always true so long as $x > 2$. Hence we can choose any $\delta > 0$, and thus $2 < x < 2 + \delta$
$\Rightarrow \left|\frac{x-2}{|x-2|} - 1\right| < \epsilon$. Thus, $\lim\limits_{x \to -2^+} \frac{x-2}{|x-2|} = 1$.

51. (a) $\lim_{x \to 400^+} \lfloor x \rfloor = 400$. Just observe that if $400 < x < 401$, then $\lfloor x \rfloor = 400$. Thus if we choose $\delta = 1$, we have for any number $\epsilon > 0$ that $400 < x < 400 + \delta \Rightarrow |\lfloor x \rfloor - 400| = |400 - 400| = 0 < \epsilon$.

(b) $\lim_{x \to 400^-} \lfloor x \rfloor = 399$. Just observe that if $399 < x < 400$ then $\lfloor x \rfloor = 399$. Thus if we choose $\delta = 1$, we have for any number $\epsilon > 0$ that $400 - \delta < x < 400 \Rightarrow |\lfloor x \rfloor - 399| = |399 - 399| = 0 < \epsilon$.

(c) Since $\lim_{x \to 400^+} \lfloor x \rfloor \neq \lim_{x \to 400^-} \lfloor x \rfloor$ we conclude that $\lim_{x \to 400} \lfloor x \rfloor$ does not exist.

52. (a) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0$; $\left|\sqrt{x} - 0\right| < \epsilon \Rightarrow -\epsilon < \sqrt{x} < \epsilon \Rightarrow 0 < x < \epsilon^2$ for x positive. Choose $\delta = \epsilon^2$ $\Rightarrow \lim_{x \to 0^+} f(x) = 0$.

(b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 \sin\left(\frac{1}{x}\right) = 0$ by the sandwich theorem since $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$ for all $x \neq 0$.
Since $|x^2 - 0| = |-x^2 - 0| = x^2 < \epsilon$ whenever $|x| < \sqrt{\epsilon}$, we choose $\delta = \sqrt{\epsilon}$ and obtain $\left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| < \epsilon$ if $-\delta < x < 0$.

(c) The function f has limit 0 at $x_0 = 0$ since both the right-hand and left-hand limits exist and equal 0.

2.5 CONTINUITY

1. No, discontinuous at $x = 2$, not defined at $x = 2$

2. No, discontinuous at $x = 3$, $1 = \lim_{x \to 3^-} g(x) \neq g(3) = 1.5$

3. Continuous on $[-1, 3]$

4. No, discontinuous at $x = 1$, $1.5 = \lim_{x \to 1^-} k(x) \neq \lim_{x \to 1^+} k(x) = 0$

5. (a) Yes (b) Yes, $\lim_{x \to -1^+} f(x) = 0$
(c) Yes (d) Yes

6. (a) Yes, $f(1) = 1$ (b) Yes, $\lim_{x \to 1} f(x) = 2$
(c) No (d) No

7. (a) No (b) No

8. $[-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3)$

9. $f(2) = 0$, since $\lim_{x \to 2^-} f(x) = -2(2) + 4 = 0 = \lim_{x \to 2^+} f(x)$

10. $f(1)$ should be changed to $2 = \lim_{x \to 1} f(x)$

11. Nonremovable discontinuity at $x = 1$ because $\lim_{x \to 1} f(x)$ fails to exist ($\lim_{x \to 1^-} f(x) = 1$ and $\lim_{x \to 1^+} f(x) = 0$).
Removable discontinuity at $x = 0$ by assigning the number $\lim_{x \to 0} f(x) = 0$ to be the value of $f(0)$ rather than $f(0) = 1$.

12. Nonremovable discontinuity at $x = 1$ because $\lim_{x \to 1} f(x)$ fails to exist ($\lim_{x \to 1^-} f(x) = 2$ and $\lim_{x \to 1^+} f(x) = 1$).
Removable discontinuity at $x = 2$ by assigning the number $\lim_{x \to 2} f(x) = 1$ to be the value of $f(2)$ rather than $f(2) = 2$.

13. Discontinuous only when $x - 2 = 0 \Rightarrow x = 2$

14. Discontinuous only when $(x+2)^2 = 0 \Rightarrow x = -2$

15. Discontinuous only when $x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 3$ or $x = 1$

16. Discontinuous only when $x^2 - 3x - 10 = 0 \Rightarrow (x-5)(x+2) = 0 \Rightarrow x = 5$ or $x = -2$

17. Continuous everywhere. ($|x-1| + \sin x$ defined for all x; limits exist and are equal to function values.)

18. Continuous everywhere. ($|x| + 1 \neq 0$ for all x; limits exist and are equal to function values.)

19. Discontinuous only at $x = 0$

20. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n-1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

21. Discontinuous when 2x is an integer multiple of π, i.e., $2x = n\pi$, n an integer $\Rightarrow x = \frac{n\pi}{2}$, n an integer, but continuous at all other x.

22. Discontinuous when $\frac{\pi x}{2}$ is an odd integer multiple of $\frac{\pi}{2}$, i.e., $\frac{\pi x}{2} = (2n-1)\frac{\pi}{2}$, n an integer $\Rightarrow x = 2n - 1$, n an integer (i.e., x is an odd integer). Continuous everywhere else.

23. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n-1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

24. Continuous everywhere since $x^4 + 1 \geq 1$ and $-1 \leq \sin x \leq 1 \Rightarrow 0 \leq \sin^2 x \leq 1 \Rightarrow 1 + \sin^2 x \geq 1$; limits exist and are equal to the function values.

25. Discontinuous when $2x + 3 < 0$ or $x < -\frac{3}{2} \Rightarrow$ continuous on the interval $\left[-\frac{3}{2}, \infty\right)$.

26. Discontinuous when $3x - 1 < 0$ or $x < \frac{1}{3} \Rightarrow$ continuous on the interval $\left[\frac{1}{3}, \infty\right)$.

27. Continuous everywhere: $(2x-1)^{1/3}$ is defined for all x; limits exist and are equal to function values.

28. Continuous everywhere: $(2-x)^{1/5}$ is defined for all x; limits exist and are equal to function values.

29. Continuous everywhere since $\lim\limits_{x \to 3} \frac{x^2 - x - 6}{x - 3} = \lim\limits_{x \to 3} \frac{(x-3)(x+2)}{x-3} = \lim\limits_{x \to 3} (x+2) = 5 = g(3)$

30. Discontinuous at $x = -2$ since $\lim\limits_{x \to -2} f(x)$ does not exist while $f(-2) = 4$.

31. $\lim\limits_{x \to \pi} \sin(x - \sin x) = \sin(\pi - \sin \pi) = \sin(\pi - 0) = \sin \pi = 0$, and function continuous at $x = \pi$.

32. $\lim\limits_{t \to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right) = \sin\left(\frac{\pi}{2}\cos(\tan(0))\right) = \sin\left(\frac{\pi}{2}\cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$, and function continuous at $t = 0$.

33. $\lim\limits_{y \to 1} \sec(y\sec^2 y - \tan^2 y - 1) = \lim\limits_{y \to 1} \sec(y\sec^2 y - \sec^2 y) = \lim\limits_{y \to 1} \sec((y-1)\sec^2 y) = \sec((1-1)\sec^2 1)$ $= \sec 0 = 1$, and function continuous at $y = 1$.

34. $\lim\limits_{x \to 0} \tan\left[\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right] = \tan\left[\frac{\pi}{4}\cos(\sin(0))\right] = \tan\left(\frac{\pi}{4}\cos(0)\right) = \tan\left(\frac{\pi}{4}\right) = 1$, and function continuous at $x = 0$.

35. $\lim_{t \to 0} \cos\left[\frac{\pi}{\sqrt{19 - 3\sec 2t}}\right] = \cos\left[\frac{\pi}{\sqrt{19 - 3\sec 0}}\right] = \cos\frac{\pi}{\sqrt{16}} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, and function continuous at $t = 0$.

36. $\lim_{x \to \frac{\pi}{6}} \sqrt{\csc^2 x + 5\sqrt{3}\tan x} = \sqrt{\csc^2\left(\frac{\pi}{6}\right) + 5\sqrt{3}\tan\left(\frac{\pi}{6}\right)} = \sqrt{4 + 5\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)} = \sqrt{9} = 3$, and function continuous at $x = \frac{\pi}{6}$.

37. $g(x) = \frac{x^2 - 9}{x - 3} = \frac{(x+3)(x-3)}{(x-3)} = x + 3,\ x \neq 3 \;\Rightarrow\; g(3) = \lim_{x \to 3}(x + 3) = 6$

38. $h(t) = \frac{t^2 + 3t - 10}{t - 2} = \frac{(t+5)(t-2)}{t-2} = t + 5,\ t \neq 2 \;\Rightarrow\; h(2) = \lim_{t \to 2}(t + 5) = 7$

39. $f(s) = \frac{s^3 - 1}{s^2 - 1} = \frac{(s^2+s+1)(s-1)}{(s+1)(s-1)} = \frac{s^2+s+1}{s+1},\ s \neq 1 \;\Rightarrow\; f(1) = \lim_{s \to 1}\left(\frac{s^2+s+1}{s+1}\right) = \frac{3}{2}$

40. $g(x) = \frac{x^2 - 16}{x^2 - 3x - 4} = \frac{(x+4)(x-4)}{(x-4)(x+1)} = \frac{x+4}{x+1},\ x \neq 4 \;\Rightarrow\; g(4) = \lim_{x \to 4}\left(\frac{x+4}{x+1}\right) = \frac{8}{5}$

41. As defined, $\lim_{x \to 3^-} f(x) = (3)^2 - 1 = 8$ and $\lim_{x \to 3^+} (2a)(3) = 6a$. For $f(x)$ to be continuous we must have $6a = 8 \;\Rightarrow\; a = \frac{4}{3}$.

42. As defined, $\lim_{x \to -2^-} g(x) = -2$ and $\lim_{x \to -2^+} g(x) = b(-2)^2 = 4b$. For $g(x)$ to be continuous we must have $4b = -2 \;\Rightarrow\; b = -\frac{1}{2}$.

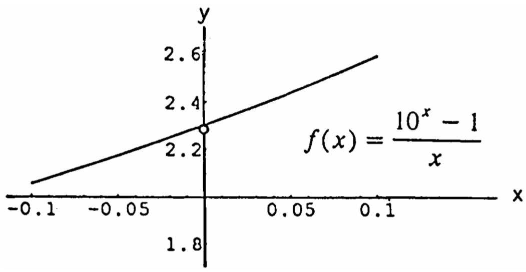
43. As defined, $\lim_{x \to 2^-} f(x) = 12$ and $\lim_{x \to 2^+} f(x) = a^2(2) - 2a = 2a^2 - 2a$. For $f(x)$ to be continuous we must have $12 = 2a^2 - 2a \;\Rightarrow\; a = 3$ or $a = -2$.

44. As defined, $\lim_{x \to 0^-} g(x) = \frac{0 - b}{b + 1} = \frac{-b}{b+1}$ and $\lim_{x \to 0^+} g(x) = (0)^2 + b = b$. For $g(x)$ to be continuous we must have $\frac{-b}{b+1} = b \;\Rightarrow\; b = 0$ or $b = -2$.

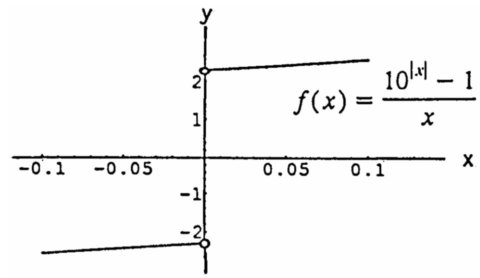
45. As defined, $\lim_{x \to -1^-} f(x) = -2$ and $\lim_{x \to -1^+} f(x) = a(-1) + b = -a + b$, and $\lim_{x \to 1^-} f(x) = a(1) + b = a + b$ and $\lim_{x \to 1^+} f(x) = 3$. For $f(x)$ to be continuous we must have $-2 = -a + b$ and $a + b = 3 \Rightarrow\; a = \frac{5}{2}$ and $b = \frac{1}{2}$.

46. As defined, $\lim_{x \to 0^-} g(x) = a(0) + 2b = 2b$ and $\lim_{x \to 0^+} g(x) = (0)^2 + 3a - b = 3a - b$, and $\lim_{x \to 2^-} g(x) = (2)^2 + 3a - b = 4 + 3a - b$ and $\lim_{x \to 0^+} g(x) = 3(2) - 5 = 1$. For $g(x)$ to be continuous we must have $2b = 3a - b$ and $4 + 3a - b = 1 \Rightarrow\; a = -\frac{3}{2}$ and $b = -\frac{3}{2}$.

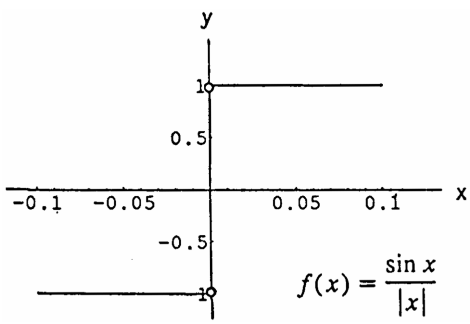
47. The function can be extended: $f(0) \approx 2.3$.



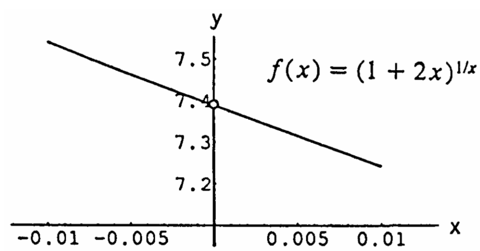
48. The function cannot be extended to be continuous at $x = 0$. If $f(0) \approx 2.3$, it will be continuous from the right. Or if $f(0) \approx -2.3$, it will be continuous from the left.



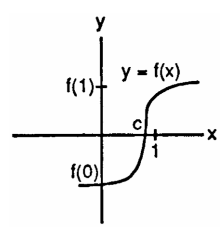
49. The function cannot be extended to be continuous at $x = 0$. If $f(0) = 1$, it will be continuous from the right. Or if $f(0) = -1$, it will be continuous from the left.



50. The function can be extended: $f(0) \approx 7.39$.



51. $f(x)$ is continuous on $[0, 1]$ and $f(0) < 0$, $f(1) > 0$ $\Rightarrow$ by the Intermediate Value Theorem $f(x)$ takes on every value between $f(0)$ and $f(1)$ $\Rightarrow$ the equation $f(x) = 0$ has at least one solution between $x = 0$ and $x = 1$.



52. $\cos x = x \Rightarrow (\cos x) - x = 0$. If $x = -\frac{\pi}{2}$, $\cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0$. If $x = \frac{\pi}{2}$, $\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} < 0$. Thus $\cos x - x = 0$ for some x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ according to the Intermediate Value Theorem, since the function $\cos x - x$ is continuous.

53. Let $f(x) = x^3 - 15x + 1$, which is continuous on $[-4, 4]$. Then $f(-4) = -3$, $f(-1) = 15$, $f(1) = -13$, and $f(4) = 5$. By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-4 < x < -1$, $-1 < x < 1$, and $1 < x < 4$. That is, $x^3 - 15x + 1 = 0$ has three solutions in $[-4, 4]$. Since a polynomial of degree 3 can have at most 3 solutions, these are the only solutions.

54. Without loss of generality, assume that $a < b$. Then $F(x) = (x - a)^2 (x - b)^2 + x$ is continuous for all values of x, so it is continuous on the interval $[a, b]$. Moreover $F(a) = a$ and $F(b) = b$. By the Intermediate Value Theorem, since $a < \frac{a+b}{2} < b$, there is a number c between a and b such that $F(x) = \frac{a+b}{2}$.

55. Answers may vary. Note that f is continuous for every value of x.
 (a) $f(0) = 10$, $f(1) = 1^3 - 8(1) + 10 = 3$. Since $3 < \pi < 10$, by the Intermediate Value Theorem, there exists a c so that $0 < c < 1$ and $f(c) = \pi$.
 (b) $f(0) = 10$, $f(-4) = (-4)^3 - 8(-4) + 10 = -22$. Since $-22 < -\sqrt{3} < 10$, by the Intermediate Value Theorem, there exists a c so that $-4 < c < 0$ and $f(c) = -\sqrt{3}$.
 (c) $f(0) = 10$, $f(1000) = (1000)^3 - 8(1000) + 10 = 999{,}992{,}010$. Since $10 < 5{,}000{,}000 < 999{,}992{,}010$, by the Intermediate Value Theorem, there exists a c so that $0 < c < 1000$ and $f(c) = 5{,}000{,}000$.

56. All five statements ask for the same information because of the intermediate value property of continuous functions.
 (a) A root of $f(x) = x^3 - 3x - 1$ is a point c where $f(c) = 0$.
 (b) The points where $y = x^3$ crosses $y = 3x + 1$ have the same y-coordinate, or $y = x^3 = 3x + 1$ $\Rightarrow f(x) = x^3 - 3x - 1 = 0$.
 (c) $x^3 - 3x = 1 \Rightarrow x^3 - 3x - 1 = 0$. The solutions to the equation are the roots of $f(x) = x^3 - 3x - 1$.
 (d) The points where $y = x^3 - 3x$ crosses $y = 1$ have common y-coordinates, or $y = x^3 - 3x = 1$ $\Rightarrow f(x) = x^3 - 3x - 1 = 0$.
 (e) The solutions of $x^3 - 3x - 1 = 0$ are those points where $f(x) = x^3 - 3x - 1$ has value 0.

57. Answers may vary. For example, $f(x) = \frac{\sin(x-2)}{x-2}$ is discontinuous at $x = 2$ because it is not defined there. However, the discontinuity can be removed because f has a limit (namely 1) as $x \to 2$.

58. Answers may vary. For example, $g(x) = \frac{1}{x+1}$ has a discontinuity at $x = -1$ because $\lim\limits_{x \to -1} g(x)$ does not exist. $\left(\lim\limits_{x \to -1^-} g(x) = -\infty \text{ and } \lim\limits_{x \to -1^+} g(x) = +\infty.\right)$

59. (a) Suppose x_0 is rational $\Rightarrow f(x_0) = 1$. Choose $\epsilon = \frac{1}{2}$. For any $\delta > 0$ there is an irrational number x (actually infinitely many) in the interval $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 0$. Then $0 < |x - x_0| < \delta$ but $|f(x) - f(x_0)| = 1 > \frac{1}{2} = \epsilon$, so $\lim\limits_{x \to x_0} f(x)$ fails to exist $\Rightarrow$ f is discontinuous at x_0 rational.
 On the other hand, x_0 irrational $\Rightarrow f(x_0) = 0$ and there is a rational number x in $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 1$. Again $\lim\limits_{x \to x_0} f(x)$ fails to exist $\Rightarrow$ f is discontinuous at x_0 irrational. That is, f is discontinuous at every point.
 (b) f is neither right-continuous nor left-continuous at any point x_0 because in every interval $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$ there exist both rational and irrational real numbers. Thus neither limits $\lim\limits_{x \to x_0^-} f(x)$ and $\lim\limits_{x \to x_0^+} f(x)$ exist by the same arguments used in part (a).

60. Yes. Both $f(x) = x$ and $g(x) = x - \frac{1}{2}$ are continuous on $[0, 1]$. However $\frac{f(x)}{g(x)}$ is undefined at $x = \frac{1}{2}$ since $g\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{f(x)}{g(x)}$ is discontinuous at $x = \frac{1}{2}$.

61. No. For instance, if $f(x) = 0$, $g(x) = \lceil x \rceil$, then $h(x) = 0(\lceil x \rceil) = 0$ is continuous at $x = 0$ and g(x) is not.

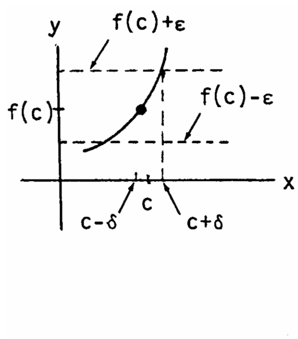
62. Let $f(x) = \frac{1}{x-1}$ and $g(x) = x + 1$. Both functions are continuous at $x = 0$. The composition $f \circ g = f(g(x)) = \frac{1}{(x+1)-1} = \frac{1}{x}$ is discontinuous at $x = 0$, since it is not defined there. Theorem 10 requires that f(x) be continuous at g(0), which is not the case here since $g(0) = 1$ and f is undefined at 1.

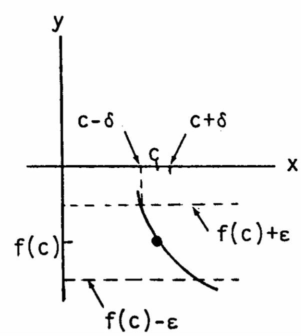
63. Yes, because of the Intermediate Value Theorem. If f(a) and f(b) did have different signs then f would have to equal zero at some point between a and b since f is continuous on [a, b].

64. Let f(x) be the new position of point x and let $d(x) = f(x) - x$. The displacement function d is negative if x is the left-hand point of the rubber band and positive if x is the right-hand point of the rubber band. By the Intermediate Value Theorem, $d(x) = 0$ for some point in between. That is, $f(x) = x$ for some point x, which is then in its original position.

65. If $f(0) = 0$ or $f(1) = 1$, we are done (i.e., $c = 0$ or $c = 1$ in those cases). Then let $f(0) = a > 0$ and $f(1) = b < 1$ because $0 \le f(x) \le 1$. Define $g(x) = f(x) - x \Rightarrow$ g is continuous on [0, 1]. Moreover, $g(0) = f(0) - 0 = a > 0$ and $g(1) = f(1) - 1 = b - 1 < 0 \Rightarrow$ by the Intermediate Value Theorem there is a number c in $(0, 1)$ such that $g(c) = 0 \Rightarrow f(c) - c = 0$ or $f(c) = c$.

66. Let $\epsilon = \frac{|f(c)|}{2} > 0$. Since f is continuous at $x = c$ there is a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$ $\Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$.
If $f(c) > 0$, then $\epsilon = \frac{1}{2} f(c) \Rightarrow \frac{1}{2} f(c) < f(x) < \frac{3}{2} f(c) \Rightarrow f(x) > 0$ on the interval $(c - \delta, c + \delta)$.
If $f(c) < 0$, then $\epsilon = -\frac{1}{2} f(c) \Rightarrow \frac{3}{2} f(c) < f(x) < \frac{1}{2} f(c) \Rightarrow f(x) < 0$ on the interval $(c - \delta, c + \delta)$.





67. By Exercises 52 in Section 2.3, we have $\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{h \to 0} f(c + h) = L$.
Thus, f(x) is continuous at $x = c \Leftrightarrow \lim_{x \to c} f(x) = f(c) \Leftrightarrow \lim_{h \to 0} f(c + h) = f(c)$.

68. By Exercise 67, it suffices to show that $\lim_{h \to 0} \sin(c + h) = \sin c$ and $\lim_{h \to 0} \cos(c + h) = \cos c$.
Now $\lim_{h \to 0} \sin(c + h) = \lim_{h \to 0} [(\sin c)(\cos h) + (\cos c)(\sin h)] = (\sin c)\left(\lim_{h \to 0} \cos h\right) + (\cos c)\left(\lim_{h \to 0} \sin h\right)$
By Example 11 Section 2.2, $\lim_{h \to 0} \cos h = 1$ and $\lim_{h \to 0} \sin h = 0$. So $\lim_{h \to 0} \sin(c + h) = \sin c$ and thus $f(x) = \sin x$ is continuous at $x = c$. Similarly,
$\lim_{h \to 0} \cos(c + h) = \lim_{h \to 0} [(\cos c)(\cos h) - (\sin c)(\sin h)] = (\cos c)\left(\lim_{h \to 0} \cos h\right) - (\sin c)\left(\lim_{h \to 0} \sin h\right) = \cos c$.
Thus, $g(x) = \cos x$ is continuous at $x = c$.

69. $x \approx 1.8794, -1.5321, -0.3473$

70. $x \approx 1.4516, -0.8547, 0.4030$

71. $x \approx 1.7549$

72. $x \approx 1.5596$

73. $x \approx 3.5156$

74. $x \approx -3.9058, 3.8392, 0.0667$

75. $x \approx 0.7391$

76. $x \approx -1.8955, 0, 1.8955$

2.6 LIMITS INVOLVING INFINITY; ASMYPTOTES OF GRAPHS

1. (a) $\lim_{x \to 2} f(x) = 0$ (b) $\lim_{x \to -3^+} f(x) = -2$
   (c) $\lim_{x \to -3^-} f(x) = 2$ (d) $\lim_{x \to 3} f(x) =$ does not exist
   (e) $\lim_{x \to 0^+} f(x) = -1$ (f) $\lim_{x \to 0^-} f(x) = +\infty$
   (g) $\lim_{x \to 0} f(x) =$ does not exist (h) $\lim_{x \to \infty} f(x) = 1$
   (i) $\lim_{x \to -\infty} f(x) = 0$

2. (a) $\lim_{x \to 4} f(x) = 2$ (b) $\lim_{x \to 2^+} f(x) = -3$
   (c) $\lim_{x \to 2^-} f(x) = 1$ (d) $\lim_{x \to 2} f(x) =$ does not exist
   (e) $\lim_{x \to -3^+} f(x) = +\infty$ (f) $\lim_{x \to -3^-} f(x) = +\infty$
   (g) $\lim_{x \to -3} f(x) = +\infty$ (h) $\lim_{x \to 0^+} f(x) = +\infty$
   (i) $\lim_{x \to 0^-} f(x) = -\infty$ (j) $\lim_{x \to 0} f(x) =$ does not exist
   (k) $\lim_{x \to \infty} f(x) = 0$ (l) $\lim_{x \to -\infty} f(x) = -1$

Note: In these exercises we use the result $\lim_{x \to \pm\infty} \frac{1}{x^{m/n}} = 0$ whenever $\frac{m}{n} > 0$. This result follows immediately from Theorem 8 and the power rule in Theorem 1: $\lim_{x \to \pm\infty} \left(\frac{1}{x^{m/n}}\right) = \lim_{x \to \pm\infty} \left(\frac{1}{x}\right)^{m/n} = \left(\lim_{x \to \pm\infty} \frac{1}{x}\right)^{m/n} = 0^{m/n} = 0.$

3. (a) -3 (b) -3

4. (a) π (b) π

5. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$

6. (a) $\frac{1}{8}$ (b) $\frac{1}{8}$

7. (a) $-\frac{5}{3}$ (b) $-\frac{5}{3}$

8. (a) $\frac{3}{4}$ (b) $\frac{3}{4}$

9. $-\frac{1}{x} \le \frac{\sin 2x}{x} \le \frac{1}{x} \Rightarrow \lim_{x \to \infty} \frac{\sin 2x}{x} = 0$ by the Sandwich Theorem

10. $-\frac{1}{3\theta} \le \frac{\cos\theta}{3\theta} \le \frac{1}{3\theta} \Rightarrow \lim_{\theta \to -\infty} \frac{\cos\theta}{3\theta} = 0$ by the Sandwich Theorem

11. $\lim_{t \to \infty} \frac{2 - t + \sin t}{t + \cos t} = \lim_{t \to \infty} \frac{\frac{2}{t} - 1 + \left(\frac{\sin t}{t}\right)}{1 + \left(\frac{\cos t}{t}\right)} = \frac{0 - 1 + 0}{1 + 0} = -1$

12. $\lim_{r \to \infty} \frac{r + \sin r}{2r + 7 - 5\sin r} = \lim_{r \to \infty} \frac{1 + \left(\frac{\sin r}{r}\right)}{2 + \frac{7}{r} - 5\left(\frac{\sin r}{r}\right)} = \lim_{r \to \infty} \frac{1 + 0}{2 + 0 - 0} = \frac{1}{2}$

13. (a) $\lim_{x \to \infty} \frac{2x + 3}{5x + 7} = \lim_{x \to \infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}} = \frac{2}{5}$ (b) $\frac{2}{5}$ (same process as part (a))

14. (a) $\lim_{x \to \infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x \to \infty} \frac{2+\left(\frac{7}{x^3}\right)}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}} = 2$

(b) 2 (same process as part (a))

15. (a) $\lim_{x \to \infty} \frac{x+1}{x^2+3} = \lim_{x \to \infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} = 0$ (b) 0 (same process as part (a))

16. (a) $\lim_{x \to \infty} \frac{3x+7}{x^2-2} = \lim_{x \to \infty} \frac{\frac{3}{x}+\frac{7}{x^2}}{1-\frac{2}{x^2}} = 0$ (b) 0 (same process as part (a))

17. (a) $\lim_{x \to \infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x \to \infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} = 7$ (b) 7 (same process as part (a))

18. (a) $\lim_{x \to \infty} \frac{1}{x^3-4x+1} = \lim_{x \to \infty} \frac{\frac{1}{x^3}}{1-\frac{4}{x^2}+\frac{1}{x^3}} = 0$ (b) 0 (same process as part (a))

19. (a) $\lim_{x \to \infty} \frac{10x^5+x^4+31}{x^6} = \lim_{x \to \infty} \frac{\frac{10}{x}+\frac{1}{x^2}+\frac{31}{x^6}}{1} = 0$ (b) 0 (same process as part (a))

20. (a) $\lim_{x \to \infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim_{x \to \infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}} = \frac{9}{2}$

(b) $\frac{9}{2}$ (same process as part (a))

21. (a) $\lim_{x \to \infty} \frac{-2x^3-2x+3}{3x^3+3x^2-5x} = \lim_{x \to \infty} \frac{-2-\frac{2}{x^2}+\frac{3}{x^3}}{3+\frac{3}{x}-\frac{5}{x^2}} = -\frac{2}{3}$

(b) $-\frac{2}{3}$ (same process as part (a))

22. (a) $\lim_{x \to \infty} \frac{-x^4}{x^4-7x^3+7x^2+9} = \lim_{x \to \infty} \frac{-1}{1-\frac{7}{x}+\frac{7}{x^2}+\frac{9}{x^4}} = -1$

(b) -1 (same process as part (a))

23. $\lim_{x \to \infty} \sqrt{\frac{8x^2-3}{2x^2+x}} = \lim_{x \to \infty} \sqrt{\frac{8-\frac{3}{x^2}}{2+\frac{1}{x}}} = \sqrt{\lim_{x \to \infty} \frac{8-\frac{3}{x^2}}{2+\frac{1}{x}}} = \sqrt{\frac{8-0}{2+0}} = \sqrt{4} = 2$

24. $\lim_{x \to -\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \lim_{x \to -\infty} \left(\frac{1+\frac{1}{x}-\frac{1}{x^2}}{8-\frac{3}{x^2}}\right)^{1/3} = \left(\lim_{x \to -\infty} \frac{1+\frac{1}{x}-\frac{1}{x^2}}{8-\frac{3}{x^2}}\right)^{1/3} = \left(\frac{1+0-0}{8-0}\right)^{1/3} = \left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$

25. $\lim_{x \to -\infty} \left(\frac{1-x^3}{x^2-7x}\right)^5 = \lim_{x \to -\infty} \left(\frac{\frac{1}{x^2}-x}{1-\frac{7}{x}}\right)^5 = \left(\lim_{x \to -\infty} \frac{\frac{1}{x^2}-x}{1-\frac{7}{x}}\right)^5 = \left(\frac{0+\infty}{1-0}\right)^5 = \infty$

26. $\lim_{x \to \infty} \sqrt{\frac{x^2-5x}{x^3+x-2}} = \lim_{x \to \infty} \sqrt{\frac{\frac{1}{x}-\frac{5}{x^2}}{1+\frac{1}{x^2}-\frac{2}{x^3}}} = \sqrt{\lim_{x \to \infty} \frac{\frac{1}{x}-\frac{5}{x^2}}{1+\frac{1}{x^2}-\frac{2}{x^3}}} = \sqrt{\frac{0-0}{1+0-0}} = \sqrt{0} = 0$

27. $\lim_{x \to \infty} \frac{2\sqrt{x}+x^{-1}}{3x-7} = \lim_{x \to \infty} \frac{\left(\frac{2}{x^{1/2}}\right)+\left(\frac{1}{x^2}\right)}{3-\frac{7}{x}} = 0$

28. $\lim_{x \to \infty} \frac{2+\sqrt{x}}{2-\sqrt{x}} = \lim_{x \to \infty} \frac{\left(\frac{2}{x^{1/2}}\right)+1}{\left(\frac{2}{x^{1/2}}\right)-1} = -1$

29. $\lim_{x \to -\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}} = \lim_{x \to -\infty} \frac{1-x^{(1/5)-(1/3)}}{1+x^{(1/5)-(1/3)}} = \lim_{x \to -\infty} \frac{1-\left(\frac{1}{x^{2/15}}\right)}{1+\left(\frac{1}{x^{2/15}}\right)} = 1$

30. $\lim_{x \to \infty} \frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}} = \lim_{x \to \infty} \frac{x+\frac{1}{x^2}}{1-\frac{1}{x}} = \infty$

31. $\lim_{x \to \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}} = \lim_{x \to \infty} \frac{2x^{1/15} - \frac{1}{x^{19/15}} + \frac{7}{x^{8/5}}}{1 + \frac{3}{x^{3/5}} + \frac{1}{x^{11/10}}} = \infty$

32. $\lim_{x \to -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4} = \lim_{x \to -\infty} \frac{\frac{1}{x^{2/3}} - 5 + \frac{3}{x}}{2 + \frac{1}{x^{1/3}} - \frac{4}{x}} = -\frac{5}{2}\sqrt[3]{x}$

33. $\lim_{x \to \infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x \to \infty} \frac{\sqrt{x^2+1}/\sqrt{x^2}}{(x+1)/\sqrt{x^2}} = \lim_{x \to \infty} \frac{\sqrt{(x^2+1)/x^2}}{(x+1)/x} = \lim_{x \to \infty} \frac{\sqrt{1+1/x^2}}{(1+1/x)} = \frac{\sqrt{1+0}}{(1+0)} = 1$

34. $\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x \to -\infty} \frac{\sqrt{x^2+1}/\sqrt{x^2}}{(x+1)/\sqrt{x^2}} = \lim_{x \to -\infty} \frac{\sqrt{(x^2+1)/x^2}}{(x+1)/(-x)} = \lim_{x \to \infty} \frac{\sqrt{1+1/x^2}}{(-1-1/x)} = \frac{\sqrt{1+0}}{(-1-0)} = -1$

35. $\lim_{x \to \infty} \frac{x-3}{\sqrt{4x^2+25}} = \lim_{x \to \infty} \frac{(x-3)/\sqrt{x^2}}{\sqrt{4x^2+25}/\sqrt{x^2}} = \lim_{x \to \infty} \frac{(x-3)/x}{\sqrt{(4x^2+25)/x^2}} = \lim_{x \to \infty} \frac{(1-3/x)}{\sqrt{4+25/x^2}} = \frac{(1-0)}{\sqrt{4+0}} = \frac{1}{2}$

36. $\lim_{x \to -\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x \to -\infty} \frac{(4-3x^3)/\sqrt{x^6}}{\sqrt{x^6+9}/\sqrt{x^6}} = \lim_{x \to -\infty} \frac{(4-3x^3)/(-x^3)}{\sqrt{(x^6+9)/x^6}} = \lim_{x \to \infty} \frac{(-4/x^3+3)}{\sqrt{1+9/x^6}} = \frac{(0+3)}{\sqrt{1+0}} = 3$

37. $\lim_{x \to 0^+} \frac{1}{3x} = \infty$ $\quad\left(\frac{\text{positive}}{\text{positive}}\right)$

38. $\lim_{x \to 0^-} \frac{5}{2x} = -\infty$ $\quad\left(\frac{\text{positive}}{\text{negative}}\right)$

39. $\lim_{x \to 2^-} \frac{3}{x-2} = -\infty$ $\quad\left(\frac{\text{positive}}{\text{negative}}\right)$

40. $\lim_{x \to 3^+} \frac{1}{x-3} = \infty$ $\quad\left(\frac{\text{positive}}{\text{positive}}\right)$

41. $\lim_{x \to -8^+} \frac{2x}{x+8} = -\infty$ $\quad\left(\frac{\text{negative}}{\text{positive}}\right)$

42. $\lim_{x \to -5^-} \frac{3x}{2x+10} = \infty$ $\quad\left(\frac{\text{negative}}{\text{negative}}\right)$

43. $\lim_{x \to 7} \frac{4}{(x-7)^2} = \infty$ $\quad\left(\frac{\text{positive}}{\text{positive}}\right)$

44. $\lim_{x \to 0} \frac{-1}{x^2(x+1)} = -\infty$ $\quad\left(\frac{\text{negative}}{\text{positive}\cdot\text{positive}}\right)$

45. (a) $\lim_{x \to 0^+} \frac{2}{3x^{1/3}} = \infty$ (b) $\lim_{x \to 0^-} \frac{2}{3x^{1/3}} = -\infty$

46. (a) $\lim_{x \to 0^+} \frac{2}{x^{1/5}} = \infty$ (b) $\lim_{x \to 0^-} \frac{2}{x^{1/5}} = -\infty$

47. $\lim_{x \to 0} \frac{4}{x^{2/5}} = \lim_{x \to 0} \frac{4}{(x^{1/5})^2} = \infty$

48. $\lim_{x \to 0} \frac{1}{x^{2/3}} = \lim_{x \to 0} \frac{1}{(x^{1/3})^2} = \infty$

49. $\lim_{x \to \left(\frac{\pi}{2}\right)^-} \tan x = \infty$

50. $\lim_{x \to \left(\frac{-\pi}{2}\right)^+} \sec x = \infty$

51. $\lim_{\theta \to 0^-} (1 + \csc\theta) = -\infty$

52. $\lim_{\theta \to 0^+} (2 - \cot\theta) = -\infty$ and $\lim_{\theta \to 0^-} (2 - \cot\theta) = \infty$, so the limit does not exist

53. (a) $\lim_{x \to 2^+} \frac{1}{x^2-4} = \lim_{x \to 2^+} \frac{1}{(x+2)(x-2)} = \infty$ $\quad\left(\frac{1}{\text{positive}\cdot\text{positive}}\right)$

(b) $\lim_{x \to 2^-} \frac{1}{x^2-4} = \lim_{x \to 2^-} \frac{1}{(x+2)(x-2)} = -\infty$ $\quad\left(\frac{1}{\text{positive}\cdot\text{negative}}\right)$

(c) $\lim_{x \to -2^+} \frac{1}{x^2-4} = \lim_{x \to -2^+} \frac{1}{(x+2)(x-2)} = -\infty$ $\quad\left(\frac{1}{\text{positive}\cdot\text{negative}}\right)$

(d) $\lim_{x \to -2^-} \frac{1}{x^2-4} = \lim_{x \to -2^-} \frac{1}{(x+2)(x-2)} = \infty$ $\quad\left(\frac{1}{\text{negative}\cdot\text{negative}}\right)$

54. (a) $\lim\limits_{x \to 1^+} \frac{x}{x^2-1} = \lim\limits_{x \to 1^+} \frac{x}{(x+1)(x-1)} = \infty$ $\left(\frac{\text{positive}}{\text{positive}\cdot\text{positive}}\right)$

(b) $\lim\limits_{x \to 1^-} \frac{x}{x^2-1} = \lim\limits_{x \to 1^-} \frac{x}{(x+1)(x-1)} = -\infty$ $\left(\frac{\text{positive}}{\text{positive}\cdot\text{negative}}\right)$

(c) $\lim\limits_{x \to -1^+} \frac{x}{x^2-1} = \lim\limits_{x \to -1^+} \frac{x}{(x+1)(x-1)} = \infty$ $\left(\frac{\text{negative}}{\text{positive}\cdot\text{negative}}\right)$

(d) $\lim\limits_{x \to -1^-} \frac{x}{x^2-1} = \lim\limits_{x \to -1^-} \frac{x}{(x+1)(x-1)} = -\infty$ $\left(\frac{\text{negative}}{\text{negative}\cdot\text{negative}}\right)$

55. (a) $\lim\limits_{x \to 0^+} \frac{x^2}{2} - \frac{1}{x} = 0 + \lim\limits_{x \to 0^+} \frac{1}{-x} = -\infty$ $\left(\frac{1}{\text{negative}}\right)$

(b) $\lim\limits_{x \to 0^-} \frac{x^2}{2} - \frac{1}{x} = 0 + \lim\limits_{x \to 0^-} \frac{1}{-x} = \infty$ $\left(\frac{1}{\text{positive}}\right)$

(c) $\lim\limits_{x \to \sqrt[3]{2}} \frac{x^2}{2} - \frac{1}{x} = \frac{2^{2/3}}{2} - \frac{1}{2^{1/3}} = 2^{-1/3} - 2^{-1/3} = 0$

(d) $\lim\limits_{x \to -1} \frac{x^2}{2} - \frac{1}{x} = \frac{1}{2} - \left(\frac{1}{-1}\right) = \frac{3}{2}$

56. (a) $\lim\limits_{x \to -2^+} \frac{x^2-1}{2x+4} = \infty$ $\left(\frac{\text{positive}}{\text{positive}}\right)$

(b) $\lim\limits_{x \to -2^-} \frac{x^2-1}{2x+4} = -\infty$ $\left(\frac{\text{positive}}{\text{negative}}\right)$

(c) $\lim\limits_{x \to 1^+} \frac{x^2-1}{2x+4} = \lim\limits_{x \to 1^+} \frac{(x+1)(x-1)}{2x+4} = \frac{2\cdot 0}{2+4} = 0$

(d) $\lim\limits_{x \to 0^-} \frac{x^2-1}{2x+4} = \frac{-1}{4}$

57. (a) $\lim\limits_{x \to 0^+} \frac{x^2-3x+2}{x^3-2x^2} = \lim\limits_{x \to 0^+} \frac{(x-2)(x-1)}{x^2(x-2)} = -\infty$ $\left(\frac{\text{negative}\cdot\text{negative}}{\text{positive}\cdot\text{negative}}\right)$

(b) $\lim\limits_{x \to 2^+} \frac{x^2-3x+2}{x^3-2x^2} = \lim\limits_{x \to 2^+} \frac{(x-2)(x-1)}{x^2(x-2)} = \lim\limits_{x \to 2^+} \frac{x-1}{x^2} = \frac{1}{4},\ x \neq 2$

(c) $\lim\limits_{x \to 2^-} \frac{x^2-3x+2}{x^3-2x^2} = \lim\limits_{x \to 2^-} \frac{(x-2)(x-1)}{x^2(x-2)} = \lim\limits_{x \to 2^-} \frac{x-1}{x^2} = \frac{1}{4},\ x \neq 2$

(d) $\lim\limits_{x \to 2} \frac{x^2-3x+2}{x^3-2x^2} = \lim\limits_{x \to 2} \frac{(x-2)(x-1)}{x^2(x-2)} = \lim\limits_{x \to 2} \frac{x-1}{x^2} = \frac{1}{4},\ x \neq 2$

(e) $\lim\limits_{x \to 0} \frac{x^2-3x+2}{x^3-2x^2} = \lim\limits_{x \to 0} \frac{(x-2)(x-1)}{x^2(x-2)} = -\infty$ $\left(\frac{\text{negative}\cdot\text{negative}}{\text{positive}\cdot\text{negative}}\right)$

58. (a) $\lim\limits_{x \to 2^+} \frac{x^2-3x+2}{x^3-4x} = \lim\limits_{x \to 2^+} \frac{(x-2)(x-1)}{x(x-2)(x+2)} = \lim\limits_{x \to 2^+} \frac{(x-1)}{x(x+2)} = \frac{1}{2(4)} = \frac{1}{8}$

(b) $\lim\limits_{x \to -2^+} \frac{x^2-3x+2}{x^3-4x} = \lim\limits_{x \to -2^+} \frac{(x-2)(x-1)}{x(x-2)(x+2)} = \lim\limits_{x \to -2^+} \frac{(x-1)}{x(x+2)} = \infty$ $\left(\frac{\text{negative}}{\text{negative}\cdot\text{positive}}\right)$

(c) $\lim\limits_{x \to 0^-} \frac{x^2-3x+2}{x^3-4x} = \lim\limits_{x \to 0^-} \frac{(x-2)(x-1)}{x(x-2)(x+2)} = \lim\limits_{x \to 0^-} \frac{(x-1)}{x(x+2)} = \infty$ $\left(\frac{\text{negative}}{\text{negative}\cdot\text{positive}}\right)$

(d) $\lim\limits_{x \to 1^+} \frac{x^2-3x+2}{x^3-4x} = \lim\limits_{x \to 1^+} \frac{(x-2)(x-1)}{x(x-2)(x+2)} = \lim\limits_{x \to 1^+} \frac{(x-1)}{x(x+2)} = \frac{0}{(1)(3)} = 0$

(e) $\lim\limits_{x \to 0^+} \frac{x-1}{x(x+2)} = -\infty$ $\left(\frac{\text{negative}}{\text{positive}\cdot\text{positive}}\right)$

and $\lim\limits_{x \to 0^-} \frac{x-1}{x(x+2)} = \infty$ $\left(\frac{\text{negative}}{\text{negative}\cdot\text{positive}}\right)$

so the function has no limit as $x \to 0$.

59. (a) $\lim\limits_{t \to 0^+} \left[2 - \frac{3}{t^{1/3}}\right] = -\infty$

(b) $\lim\limits_{t \to 0^-} \left[2 - \frac{3}{t^{1/3}}\right] = \infty$

60. (a) $\lim\limits_{t \to 0^+} \left[\frac{1}{t^{3/5}} + 7\right] = \infty$

(b) $\lim\limits_{t \to 0^-} \left[\frac{1}{t^{3/5}} + 7\right] = -\infty$

61. (a) $\lim\limits_{x \to 0^+} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$

(b) $\lim\limits_{x \to 0^-} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$

(c) $\lim\limits_{x \to 1^+} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$

(d) $\lim\limits_{x \to 1^-} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$

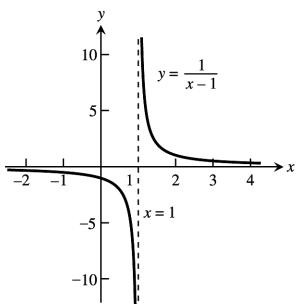
62. (a) $\lim\limits_{x \to 0^+} \left[\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right] = \infty$

(b) $\lim\limits_{x \to 0^-} \left[\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right] = -\infty$

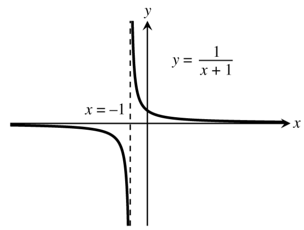
(c) $\lim\limits_{x \to 1^+} \left[\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right] = -\infty$

(d) $\lim\limits_{x \to 1^-} \left[\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right] = -\infty$

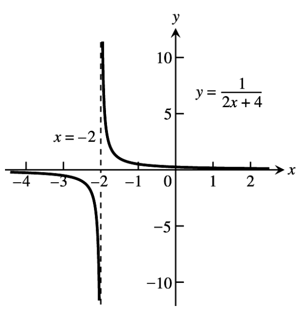
63. $y = \frac{1}{x-1}$



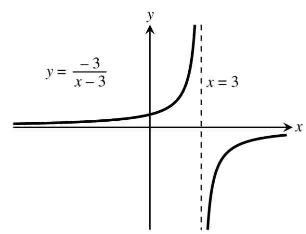
64. $y = \frac{1}{x+1}$



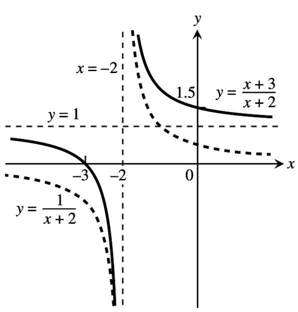
65. $y = \frac{1}{2x+4}$



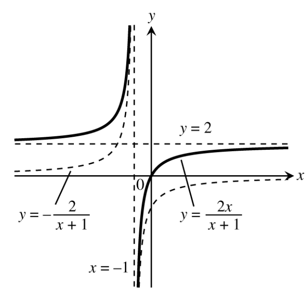
66. $y = \frac{-3}{x-3}$



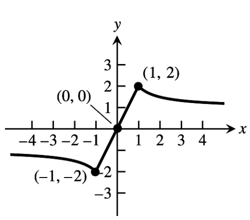
67. $y = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$



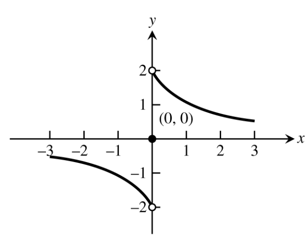
68. $y = \frac{2x}{x+1} = 2 - \frac{2}{x+1}$



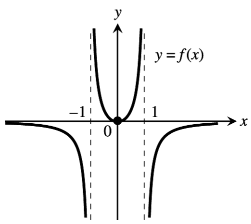
69. Here is one possibility.



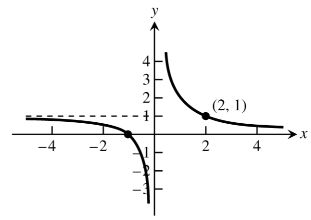
70. Here is one possibility.



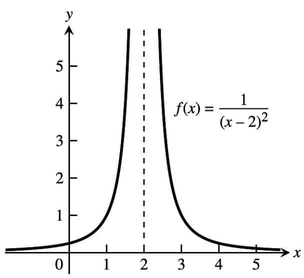
71. Here is one possibility.



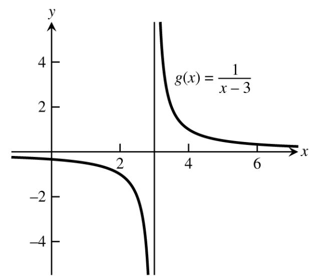
72. Here is one possibility.



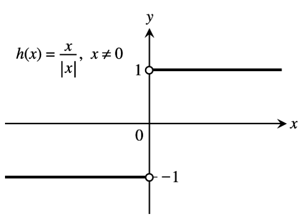
73. Here is one possibility.



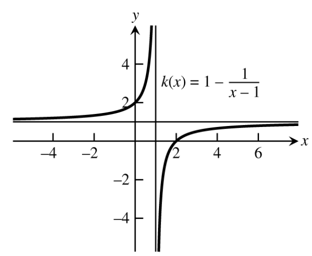
74. Here is one possibility.



75. Here is one possibility.



76. Here is one possibility.



77. Yes. If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 2$ then the ratio of the polynomials' leading coefficients is 2, so $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = 2$ as well.

78. Yes, it can have a horizontal or oblique asymptote.

79. At most 1 horizontal asymptote: If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$, then the ratio of the polynomials' leading coefficients is L, so $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = L$ as well.

80. $$\lim_{x \to \infty} \left(\sqrt{x+9} - \sqrt{x+4}\right) = \lim_{x \to \infty} \left[\sqrt{x+9} - \sqrt{x+4}\right] \cdot \left[\frac{\sqrt{x+9}+\sqrt{x+4}}{\sqrt{x+9}+\sqrt{x+4}}\right] = \lim_{x \to \infty} \frac{(x+9)-(x+4)}{\sqrt{x+9}+\sqrt{x+4}}$$
$$= \lim_{x \to \infty} \frac{5}{\sqrt{x+9}+\sqrt{x+4}} = \lim_{x \to \infty} \frac{\frac{5}{\sqrt{x}}}{\sqrt{1+\frac{9}{x}}+\sqrt{1+\frac{4}{x}}} = \frac{0}{1+1} = 0$$

81. $$\lim_{x \to \infty} \left(\sqrt{x^2+25} - \sqrt{x^2-1}\right) = \lim_{x \to \infty} \left[\sqrt{x^2+25} - \sqrt{x^2-1}\right] \cdot \left[\frac{\sqrt{x^2+25}+\sqrt{x^2-1}}{\sqrt{x^2+25}+\sqrt{x^2-1}}\right] = \lim_{x \to \infty} \frac{(x^2+25)-(x^2-1)}{\sqrt{x^2+25}+\sqrt{x^2-1}}$$
$$= \lim_{x \to \infty} \frac{26}{\sqrt{x^2+25}+\sqrt{x^2-1}} = \lim_{x \to \infty} \frac{\frac{26}{x}}{\sqrt{1+\frac{25}{x^2}}+\sqrt{1-\frac{1}{x^2}}} = \frac{0}{1+1} = 0$$

82. $$\lim_{x \to -\infty} \left(\sqrt{x^2+3} + x\right) = \lim_{x \to -\infty} \left[\sqrt{x^2+3} + x\right] \cdot \left[\frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x}\right] = \lim_{x \to -\infty} \frac{(x^2+3)-(x^2)}{\sqrt{x^2+3}-x}$$
$$= \lim_{x \to -\infty} \frac{3}{\sqrt{x^2+3}-x} = \lim_{x \to -\infty} \frac{\frac{3}{\sqrt{x^2}}}{\sqrt{1+\frac{3}{x^2}}-\frac{x}{\sqrt{x^2}}} = \lim_{x \to -\infty} \frac{-\frac{3}{x}}{\sqrt{1+\frac{3}{x^2}}+1} = \frac{0}{1+1} = 0$$

83. $\lim\limits_{x \to -\infty} \left(2x + \sqrt{4x^2 + 3x - 2}\right) = \lim\limits_{x \to -\infty} \left[2x + \sqrt{4x^2 + 3x - 2}\right] \cdot \left[\frac{2x - \sqrt{4x^2+3x-2}}{2x - \sqrt{4x^2+3x-2}}\right] = \lim\limits_{x \to -\infty} \frac{(4x^2) - (4x^2+3x-2)}{2x - \sqrt{4x^2+3x-2}}$
$= \lim\limits_{x \to -\infty} \frac{-3x+2}{2x - \sqrt{4x^2+3x-2}} = \lim\limits_{x \to -\infty} \frac{\frac{-3x+2}{\sqrt{x^2}}}{\frac{2x}{\sqrt{x^2}} - \sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}} = \lim\limits_{x \to -\infty} \frac{\frac{-3x+2}{-x}}{\frac{2x}{-x} - \sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}} = \lim\limits_{x \to -\infty} \frac{3 - \frac{2}{x}}{-2 - \sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}}$
$= \frac{3-0}{-2-2} = -\frac{3}{4}$

84. $\lim\limits_{x \to \infty} \left(\sqrt{9x^2 - x} - 3x\right) = = \lim\limits_{x \to \infty} \left[\sqrt{9x^2 - x} - 3x\right] \cdot \left[\frac{\sqrt{9x^2-x}+3x}{\sqrt{9x^2-x}+3x}\right] = \lim\limits_{x \to \infty} \frac{(9x^2-x)-(9x^2)}{\sqrt{9x^2-x}+3x}$
$= \lim\limits_{x \to \infty} \frac{-x}{\sqrt{9x^2-x}+3x} = \lim\limits_{x \to \infty} \frac{-\frac{x}{x}}{\sqrt{\frac{9x^2}{x^2} - \frac{x}{x^2}} + \frac{3x}{x}} = \lim\limits_{x \to \infty} \frac{-1}{\sqrt{9 - \frac{1}{x}} + 3} = \frac{-1}{3+3} = -\frac{1}{6}$

85. $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x}\right) = \lim\limits_{x \to \infty} \left[\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x}\right] \cdot \left[\frac{\sqrt{x^2+3x}+\sqrt{x^2-2x}}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}\right] = \lim\limits_{x \to \infty} \frac{(x^2+3x)-(x^2-2x)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$
$= \lim\limits_{x \to \infty} \frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} = \lim\limits_{x \to \infty} \frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}} = \frac{5}{1+1} = \frac{5}{2}$

86. $\lim\limits_{x \to \infty} \sqrt{x^2 + x} - \sqrt{x^2 - x} = \lim\limits_{x \to \infty} \left[\sqrt{x^2 + x} - \sqrt{x^2 - x}\right] \cdot \left[\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}\right] = \lim\limits_{x \to \infty} \frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}$
$= \lim\limits_{x \to \infty} \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}} = \lim\limits_{x \to \infty} \frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}} = \frac{2}{1+1} = 1$

87. For any $\epsilon > 0$, take $N = 1$. Then for all $x > N$ we have that $|f(x) - k| = |k - k| = 0 < \epsilon$.

88. For any $\epsilon > 0$, take $N = 1$. Then for all $y < -N$ we have that $|f(x) - k| = |k - k| = 0 < \epsilon$.

89. For every real number $-B < 0$, we must find a $\delta > 0$ such that for all x, $0 < |x - 0| < \delta \Rightarrow \frac{-1}{x^2} < -B$. Now, $-\frac{1}{x^2} < -B < 0 \Leftrightarrow \frac{1}{x^2} > B > 0 \Leftrightarrow x^2 < \frac{1}{B} \Leftrightarrow |x| < \frac{1}{\sqrt{B}}$. Choose $\delta = \frac{1}{\sqrt{B}}$, then $0 < |x| < \delta \Rightarrow |x| < \frac{1}{\sqrt{B}}$ $\Rightarrow \frac{-1}{x^2} < -B$ so that $\lim\limits_{x \to 0} -\frac{1}{x^2} = -\infty$.

90. For every real number $B > 0$, we must find a $\delta > 0$ such that for all x, $0 < |x - 0| < \delta \Rightarrow \frac{1}{|x|} > B$. Now, $\frac{1}{|x|} > B > 0 \Leftrightarrow |x| < \frac{1}{B}$. Choose $\delta = \frac{1}{B}$. Then $0 < |x - 0| < \delta \Rightarrow |x| < \frac{1}{B} \Rightarrow \frac{1}{|x|} > B$ so that $\lim\limits_{x \to 0} \frac{1}{|x|} = \infty$.

91. For every real number $-B < 0$, we must find a $\delta > 0$ such that for all x, $0 < |x - 3| < \delta \Rightarrow \frac{-2}{(x-3)^2} < -B$. Now, $\frac{-2}{(x-3)^2} < -B < 0 \Leftrightarrow \frac{2}{(x-3)^2} > B > 0 \Leftrightarrow \frac{(x-3)^2}{2} < \frac{1}{B} \Leftrightarrow (x-3)^2 < \frac{2}{B} \Leftrightarrow 0 < |x - 3| < \sqrt{\frac{2}{B}}$. Choose $\delta = \sqrt{\frac{2}{B}}$, then $0 < |x - 3| < \delta \Rightarrow \frac{-2}{(x-3)^2} < -B < 0$ so that $\lim\limits_{x \to 3} \frac{-2}{(x-3)^2} = -\infty$.

92. For every real number $B > 0$, we must find a $\delta > 0$ such that for all x, $0 < |x - (-5)| < \delta \Rightarrow \frac{1}{(x+5)^2} > B$. Now, $\frac{1}{(x+5)^2} > B > 0 \Leftrightarrow (x+5)^2 < \frac{1}{B} \Leftrightarrow |x + 5| < \frac{1}{\sqrt{B}}$. Choose $\delta = \frac{1}{\sqrt{B}}$. Then $0 < |x - (-5)| < \delta$ $\Rightarrow |x + 5| < \frac{1}{\sqrt{B}} \Rightarrow \frac{1}{(x+5)^2} > B$ so that $\lim\limits_{x \to -5} \frac{1}{(x+5)^2} = \infty$.

93. (a) We say that f(x) approaches infinity as x approaches x_0 from the left, and write $\lim\limits_{x \to x_0^-} f(x) = \infty$, if for every positive number B, there exists a corresponding number $\delta > 0$ such that for all x, $x_0 - \delta < x < x_0 \Rightarrow f(x) > B$.
(b) We say that f(x) approaches minus infinity as x approaches x_0 from the right, and write $\lim\limits_{x \to x_0^+} f(x) = -\infty$, if for every positive number B (or negative number $-B$) there exists a corresponding number $\delta > 0$ such that for all x, $x_0 < x < x_0 + \delta \Rightarrow f(x) < -B$.